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Distributions of Soft Binary Piecewise Operations over Complementary Soft Binary Piecewise Star (∗) and Theta (θ) Operations

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ABSTRACT

Soft set theory deals with uncertainty and it has been applied to many fields both as a theoretical and application aspect. Since its inception, different kinds of soft set operations have been defined and used in various types. This paper is a theoretical study of soft sets and in this paper, it is aimed to contribute to the soft set literature by obtaining the distributions of different kinds of soft binary piecewise operations over complementary soft binary piecewise star and theta operations in order to obtain some algebraic structures.

Esnek İkili Parçalı İşlemlerin Tümleyenli Esnek İkili Parçalı Yıldız (∗) ve Teta (θ) İşlemleri Üzerine Dağılması

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Koşullu tümleyenler
Yıldız işlemi
Teta işlemi

ÖZ

Esnek küme teorisi belirsizliklerle ilgilenir ve hem teorik hem de uygulama yönüyle birçok alana uygulanmıştır. Başlangıcından bu yana, farklı çeşitlerde esnek küme işlemleri tanımlanmış ve çeşitli türlerde kullanılmıştır. Bu makale esnek kümelerin teorik bir çalışmasıdır ve bu çalışmada, bazı cebirsel yapılar elde etmek için tümleyenli esnek ikili parçalı yıldız ve teta işlemleri üzerine esnek ikili parçalı işlemlerin dağılımlarının elde edilmesiyle esnek küme literatürüne katkıda bulunulması amaçlanmıştır.

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1. Introduction

Molodtsov (1999) introduced Soft Set Theory as a mathematical tool to overcome uncertainties. This theory has been applied to many fields by Özlü (2022a, 2022b) and Paik and Mondal (2022). As regards algebraic structures, it has been implemented by Atagün and Aygün

(2016) and Addis et al. (2022). Riaz and Hashimi (2019) and Ayub et al. (2021) studied on Linear Diophantine Fuzzy Sets, Riaz et al. (2023) on Linear Diophantine Fuzzy aggregation operators, Riaz et al. (2021) on Spherical Linear Diophantine Fuzzy Sets, and they are all top recent topics as novel mathematical approaches to model vagueness and uncertainty in decision-making problems. Maji et al. (2003) and Pei and Miao (2005) were the first to study on soft set operations. After then, Ali et al. (2009) introduced several soft set operations (restricted and extended soft set operations) and Sezgin and Atagün (2011) examined their basic properites. Sezgin et al. (2019) introduced a new soft set operation called extended difference of soft sets and Stojanovic (2021) defined extended symmetric difference of soft sets and investigated its properties.

Çağman (2021) introduced two conditional complements of sets as (inclusive complement and exclusive complement) and the relationships between them were explored. After this study, Sezgin et al. (2023c) defined new complements. They also transferred these complements to soft set theory, and Aybek (2023) introduced some new restricted soft set operations and extended soft set operations. By changing the form of extended soft set operations using the complement at the first and second row of the piecewise function of extended soft set operations, Demirci (2024), Sarıalioğlu (2024) and Akbulut (2024) proposed a new type of soft set operation and studied their basic properties. Also, Eren (2019) defined a new type of soft difference operations and by being inspired this study, Yavuz (2024) and Sezgin and Yavuz (2023a) defined some new soft set operations, which they call soft binary piecewise operations and they studied their basic properties. Also, Sezgin and Demirci (2023), Sezgin and Atagün (2023), Sezgin and Yavuz (2023b), Sezgin and Aybek (2023), Sezgin et al. (2023a, 2023b), Sezgin and Dagtoros (2023) continued their work on soft set operations by defining a new type of soft binary piecewise operation by changing the form of soft binary piecewise operation using the complement at the first row of the soft binary piecewise operations.

Sezgin and Demirci (2023) and Sezgin and Sarıalioğlu (2024) defined complementary soft binary piecewise theta and complementary soft binary piecewise star operation, respectively. The algebraic properties of these new operations were examined in detail. Especially the distributions of these operations over extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, complementary soft binary piecewise operations and restricted soft set operations were handled . In this study, with Section 3, we aim to contribute to the literature of soft set theory by obtaining the distributions of soft binary piecewise operations over complementary soft binary piecewise star and theta operations in order to acquire some algebraic structures.

2. Preliminaries

Definition 2.1. Let U be the universal set, E be the parameter set, P(U) be the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U where F is a set-valued function such that $F: A \rightarrow P(U)$. (Molodtsov, 1999)

The set of all the soft sets over U is designated by $S_E(U)$, and throughout this paper, all the soft sets are the elements of $S_E(U)$.

Çağman (2021) defined two conditional complements of sets, for the ease of illustration, we show these complements as + and θ, respectively. These complements are defined as following: Let A and B be two subsets of U. B-inclusive complement of A is defined by, A+B=A'∪B and B-exclusive complement of A is defined by AθB=A'∩B'. Here, U refers to a universe, A' is the complement of A over U. Sezgin et al. (2023c) introduced such new three complements as binary operations of sets as following: Let A and B be two subsets of U. Then, A*B=A'∪B', AγB=A'∩B, A$\boldsymbol{\lambda}$B=A∪B'. Aybek (2024) conveyed these classical sets to soft sets, and they defined restricted and extended soft set operations and examined their properties.

As a summary for soft set operations, we can categorize all types of soft set operations as following: Let "∇" be used to represent the set operations (i.e., here ∇ can be ∩,∪,\, Δ, +,θ, *, λ,γ), then restricted operations, extended operations, complementary extended operations, soft binary piecewise operations, complementary soft binary piecewise operations are defined in soft set theory as following:

Definition 2.2. Let (D, K) and (J, R) be soft sets over U. The restricted ∇ operation of (D, K) and (J, R) is the soft set (Y,S), denoted by, $(D, K)\nabla_R(J, R) = (Y, S)$, where $S = K \cap R \neq \emptyset$ and $\forall s \in S$, $Y(s) = D(s)\nabla J(s)$. (Ali et. al., 2009; Sezgin and Atagün, 2011; Aybek, 2024)

Definition 2.3. Let (D, K) and (J,R) be soft sets over U. The extended ∇ operation of (D, K) and (J, R) is the soft set (Y,S), denoted by, $(D, K)\nabla_\varepsilon(J, R) = (Y, S)$, where $S = K \cup R$ and $\forall s \in S$,

$$Y(s) = \begin{cases} D(s), & s \in K\backslash R, \\ J(s), & s \in R\backslash K, \\ D(s)\nabla J(s), & s \in K \cap R. \end{cases}$$

(Maji et al., 2003; Ali et al., 2009; Sezgin et al., 2019; Stojanovic, 2021; Aybek, 2024)

Definition 2.4. Let (D, K) and (J, R) be soft sets over U. The complementary extended ∇

operation of (D,K) and (J,R) is the soft set (Y,S), denoted by, $(\mathrm{D},\mathrm{K}) \overset{*}{\nabla_{\varepsilon}} (\mathrm{J},\mathrm{R}) = (\mathrm{Y},\mathrm{S})$, where $\mathrm{S} = \mathrm{K} \cup \mathrm{R}$ and $\forall \mathrm{s} \in \mathrm{S}$,

$$\mathrm{Y}(\mathrm{s}) = \begin{cases} \mathrm{D}'(\mathrm{s}), & \mathrm{s} \in \mathrm{K}\backslash\mathrm{R} \\ \mathrm{J}'(\mathrm{s}), & \mathrm{s} \in \mathrm{R}\backslash\mathrm{K}, \\ \mathrm{D}(\mathrm{s})\ \nabla \mathrm{J}(\mathrm{s}), & \mathrm{s} \in \mathrm{K} \cap \mathrm{R}. \end{cases}$$

(Sarıalioğlu, 2024; Demirci, 2024; Akbulut, 2024)

Definition 2.5. Let (D,K) and (J,R) be soft sets over U. The soft binary piecewise ∇ operation of (D,K) and (J,R) is the soft set (Y,K), denoted by, $(\mathrm{D},\mathrm{K}) \underset{\nabla}{\sim} (\mathrm{J},\mathrm{R}) = (\mathrm{Y},\mathrm{P})$, where $\forall$s∈K,

$$\mathrm{Y}(\mathrm{s}) = \begin{cases} \mathrm{D}(\mathrm{s}), & \mathrm{s} \in \mathrm{K}\backslash\mathrm{R} \\ \mathrm{D}(\mathrm{s})\nabla\mathrm{J}(\mathrm{s}), & \mathrm{s} \in \mathrm{K}\cap\mathrm{R} \end{cases}$$

(Eren, 2019; Yavuz, 2024, Sezgin and Yavuz, 2023a)

Definition 2.6. Let $(\mathrm{D},\mathrm{K})$and (J,R)be soft sets over U. The complementary soft binary piecewise ∇ operation of (D,K) and (J,R) is the soft set (Y,K), denoted by, $(\mathrm{D},\mathrm{K}) \overset{*}{\underset{\nabla}{\sim}} (\mathrm{J},\mathrm{R}) = (\mathrm{Y},\mathrm{K})$, where $\forall$s∈K;

$$\mathrm{Y}(\mathrm{s}) = \begin{cases} \mathrm{D}'(\mathrm{s}), & \mathrm{s} \in \mathrm{K}\backslash\mathrm{R} \\ \mathrm{D}(\mathrm{s})\nabla\mathrm{J}(\mathrm{s}), & \mathrm{s} \in \mathrm{K}\cap\mathrm{R} \end{cases}$$

(Sezgin and Demirci, 2023; Sezgin and Atagün, 2023; Sezgin and Aybek, 2023; Sezgin et al., 2023a; Sezgin et al., 2023b; Sezgin and Yavuz, 2023b; Sezgin and Dagtoros, 2023; Sezgin and Çağman, 2024; Sezgin and Sarıalioğlu, 2024)

Definition 2.7. Let (A,F) and (B,G) be soft sets over U. The complementary soft binary piecewise star operation of (A,F) and (B,G) is the soft set (C,F), denoted by, $(\mathrm{A},\mathrm{F}) \overset{*}{\underset{*}{\sim}} (\mathrm{B},\mathrm{G}) = (\mathrm{C},\mathrm{F})$, where $\forall$t∈F,

$$\mathrm{C}(\mathrm{s}) = \begin{cases} \mathrm{A}'(\mathrm{s}), & \mathrm{s} \in \mathrm{F}\backslash\mathrm{G} \\ \mathrm{A}'(\mathrm{s})\cup\mathrm{B}'(\mathrm{s} & \mathrm{s} \in \mathrm{F}\cap\mathrm{G} \end{cases}$$

(Sezgin and Demirci, 2023)

Definition 2.8. Let (A, F) and (B, G) be soft sets over U. The complementary soft binary piecewise theta operation of (A, F) and (B, G) is the soft set (C,F), denoted by, $(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)=(C,F)$, where ∀t∈F,

$$C(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

(Sezgin and Sarıalioğlu, 2024)

3. Distribution of Soft Binary Piecewise Operations Over Complementary Soft Binary Piecewise Star and Theta Operations

3.1.1. Distribution of soft binary piecewise operations over complementary soft binary piecewise star operation:

1) $(A,F)\underset{+}{\sim}[(B,G)\overset{*}{\underset{*}{\sim}}(C,H)]=[(A,F)\underset{*}{\sim}(B,G)]\tilde{\cup}[(C,H)\underset{*}{\sim}(A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\overset{*}{\underset{*}{\sim}}(C,H)=(M,G)$, where ∀s∈G;

$$M(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\underset{+}{\sim}(M,G)=(N,F)$, where ∀s∈F;

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A'(s)\cup M(s), & s\in F\cap G \end{cases}$$

Thus,

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A'(s)\cup B'(s), & s\in F\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cup[B'(s)\cup C'(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{*}{\sim}(B,G)]\tilde{\cup}[(C,H)\underset{*}{\sim}(A,F)]$. Let $(A,F)\underset{*}{\sim}(B,G)=(V,F)$, where ∀s∈F;

$$\begin{cases} A(s), & s\in F\setminus G \\ & \end{cases}$$

$V(s)=$

$A'(s) \cup B'(s), \quad s \in F \cap G$

Suppose that $(C,H) \underset{*}{\sim} (A,F)=(W,H)$, where $\forall s \in H$;

$$W(s)=\begin{cases} C(s), & s \in H \setminus F \\ C'(s) \cup A'(s), & s \in H \cap F \end{cases}$$

Let $(V,F)\ \tilde{\cup}\ (W,H)=(T,F)$ $\forall s \in F$;

$$T(s)=\begin{cases} V(s), & s \in F \setminus H \\ V(s) \cup W(s), & s \in F \cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s \in (F \setminus G) \setminus H = F \cap G' \cap H' \\ A'(s) \cup B'(s), & s \in (F \cap G) \setminus H = F \cap G \cap H' \\ A(s) \cup C(s), & s \in (F \setminus G) \cap (H \setminus F) = \emptyset \\ A(s) \cup [C'(s) \cup A'(s)], & s \in (F \setminus G) \cap (H \cap F) = F \cap G' \cap H \\ [A'(s) \cup B'(s)] \cup C(s), & s \in (F \cap G) \cap (H \setminus F) = \emptyset \\ [A'(s) \cup B'(s)] \cup [C'(s) \cup A'(s)], & s \in (F \cap G) \cap (H \cap F) = F \cap G \cap H \end{cases}$$

Since $F \setminus G = F \cap G'$, if $s \in G'$, then $s \in H \setminus G$ or $s \in (G \cup H)'$. Hence, if $s \in F \setminus G$, $s \in F \cap G' \cap H'$ or $s \in F \cap G' \cap H$. Thus, it is seen that $(N,F)=(T,F)$.

2) $[(A,F) \overset{*}{\underset{*}{\sim}} (B,G)]\ \tilde{\mp}\ (C,H)=[(A,F) \overset{*}{\underset{\cup}{\sim}} (C,H)]\ \tilde{\cap}\ [(B,G)\ \tilde{\cup}\ (C,H)]$, where $F \cap G \cap H' = \emptyset$.

Proof: Let $(A,F) \overset{*}{\underset{*}{\sim}} (B,G)=(M,F)$, where $\forall s \in F$;

$$M(s)=\begin{cases} A'(s), & s \in F \setminus G \\ A'(s) \cup B'(s), & s \in F \cap G \end{cases}$$

Let $(M,F)\ \tilde{\mp}\ (C,H)=(N,F)$, where $\forall s \in F$;

$$N(s)=\begin{cases} M(s), & s \in F \setminus H \\ M'(s) \cup C(s), & s \in F \cap H \end{cases}$$

Thus,

$$
N(s)=\begin{cases}
A'(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\
A'(s)\cup B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\
A(s)\cup C(s), & s\in(F\backslash G)\cap H=F\cap G'\cap H \\
[A(s)\cap B(s)]\cup C(s), & s\in(F\cap G)\cap H=F\cap G\cap H
\end{cases}
$$

Now let's handle $[(A,F)\underset{\cup}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\tilde{\cup}(C,H)$. Let $(A,F)\underset{\cup}{\overset{*}{\sim}}(C,H)=(V,F)$, where

$\forall s\in F$;

$$
V(s)=\begin{cases}
A'(s), & s\in F\backslash H \\
A(s)\cup C(s), & s\in F\cap H
\end{cases}
$$

Suppose that $(B,G)\tilde{\cup}(C,H)=(W,G)$, where $\forall s\in G$;

$$
W(s)=\begin{cases}
B(s), & s\in G\backslash H \\
B(s)\cup C(s), & s\in G\cap H
\end{cases}
$$

Let $(V,F)\tilde{\cap}\,(W,H)=(T,F)$, where $\forall s\in F$;

$$
T(s)=\begin{cases}
V(s), & s\in F\backslash H \\
V(s)\cap W(s), & s\in F\cap H
\end{cases}
$$

Thus,

$$
T(s)=\begin{cases}
A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\
A(s)\cup C(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\
A'(s)\cap B(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\
A'(s)\cap[B(s)\cup C(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\
[A(s)\cup C(s)]\cap B(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\
[A(s)\cup C(s)]\cap[B(s)\cup C(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H
\end{cases}
$$

It is seen that $(N,F)=(T,F)$.

3) $(A,F)\tilde{\gamma}[(B,G)\underset{*}{\overset{*}{\sim}}(C,H)]=[(A,F)\underset{\theta}{\sim}(B,G)]\tilde{\cup}[(C,H)\underset{\theta}{\sim}(A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\underset{*}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{\gamma}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\backslash G \\ A'(s)\cap M(s), & s\in F\cap G \end{cases}$$

Thus,

$$N(s)=\begin{cases} A(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cap[(B'(s)\cup C'(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\tilde{\theta}(B,G)]\tilde{\cup}[(C,H)\tilde{\theta}(A,F)]$. Let $(A,F)\tilde{\theta}(B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\tilde{\theta}(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\backslash F \\ C'(s)\cap A'(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cup C(s), & s\in(F\backslash G)\cap(H\backslash F)=\emptyset \\ A(s)\cup[C'(s)\cap A'(s)], & s\in(F\backslash G)\cap(H\cap F)=F\cap G'\cap H \\ [A'(s)\cap B'(s)]\cup C(s), & s\in(F\cap G)\cap(H\backslash F)=\emptyset \\ [A'(s)\cap B'(s)]\cup[C'(s)\cap A'(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

Since F\G=F∩G’, if s∈G’, then s∈H\G or s∈(G∪H)’. Hence, if s∈F\G, s∈F∩G’∩H’ or s∈F∩G’∩H. Thus, it is seen that (N,F)=(T,F).

4) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\tilde{\gamma}(C,H)=[(A,F)\underset{\cap}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{\cap}{\overset{*}{\sim}}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\theta}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M'(s)\cap C(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cap C(s) & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A(s)\cap B(s)]\cap C(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let’s handle $[(A,F)\underset{\cap}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{\cap}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\cap}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A(s)\cap C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\cap}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B(s)\cap C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\ A(s)\cap C(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cap B'(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cap[B(s)\cap C(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\ [A(s)\cap C(s)]\cap B'(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\ [A(s)\cap C(s)]\cap[B(s)\cap C(s) & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that (N,F)=(T,F).

5) $(A,F)\tilde{\backslash}[(B,G)\overset{*}{\underset{*}{\sim}}(C,H)]=[(A,F)\tilde{\cap}(B,G)]\tilde{\cap}[(C,H)\tilde{\cap}(A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\overset{*}{\underset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{\backslash}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\backslash G \\ A(s)\cap M'(s), & s\in F\cap G \end{cases}$$

Thus,

$$N(s)=\begin{cases} A(s), & s\in F\backslash G \\ A(s)\cap B(s), & s\in F\cap(G\backslash H)=F\cap G\cap H' \\ A(s)\cap[(B(s)\cap C(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\tilde{\cap}(B,G)]\tilde{\cap}[(C,H)\tilde{\cap}(A,F)]$. Let $(A,F)\tilde{\cap}(B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\backslash G \\ A(s)\cap B(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\tilde{\cap}(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\backslash F \\ C(s)\cap A(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A(s)\cap B(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cap C(s), & s\in (F\setminus G)\cap (H\setminus F)=\emptyset \\ A(s)\cap [C(s)\cap A(s)], & s\in (F\setminus G)\cap (H\cap F)=F\cap G'\cap H \\ [A(s)\cap B(s)]\cap C(s), & s\in (F\cap G)\cap (H\setminus F)=\emptyset \\ [A(s)\cap B(s)]\cap [C(s)\cap A(s)], & s\in (F\cap G)\cap (H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\setminus G=F\cap G'$, if $s\in G'$, then $s\in H\setminus G$ or $s\in (G\cup H)'$. Hence, if $s\in F\setminus G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$. Thus, it is seen that $(N,F)=(T,F)$.

6) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\tilde{\setminus}(C,H)=[(A,F)\underset{\theta}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]$.

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\setminus}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A'(s)\cap C'(s), & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A'(s)\cup B'(s)]\cap C'(s), & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\theta}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A'(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall$ $s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap H' \\ A'(s)\cap C'(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cup B'(s), & s\in (F\setminus H)\cap (G\setminus H)=F\cap G\cap H' \\ A'(s)\cup [B'(s)\cap C'(s)], & s\in (F\setminus H)\cap (G\cap H)=\emptyset \\ [A'(s)\cap C'(s)]\cup B'(s), & s\in (F\cap H)\cap (G\setminus H)=\emptyset \\ [A'(s)\cap C'(s)]\cup [B'(s)\cap C'(s)], & s\in (F\cap H)\cap (G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

7) $(A,F)\tilde{\lambda}[(B,G)\underset{*}{\overset{*}{\sim}}(C,H)]=[(A,F)\ \tilde{\cup}\ (B,G)]\tilde{\cap}[(C,H)\ \tilde{\cup}\ (A,F)]$.

Proof: Let $(B,G)\underset{*}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{\lambda}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cup M'(s), & s\in F\cap G \end{cases}$$

Hence,

$$N(s)=\begin{cases} A(s), & s\in F\backslash G \\ A(s)\cup B(s), & s\in F\cap(G\backslash H)=F\cap G\cap H' \\ A(s)\cup[(B(s)\cap C(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\,\tilde{\cup}\,(B,G)]\tilde{\cap}[(C,H)\,\tilde{\cup}(A,F)]$. Let $(A,F)\,\tilde{\cup}\,(B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\backslash G \\ A(s)\cup B(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\tilde{\cup}\,(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\backslash F \\ C(s)\cup A(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$ $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Hence

$$T(s)=\begin{cases} A(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A(s)\cup B(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cap C(s), & s\in(F\backslash G)\cap(H\backslash F)=\emptyset \\ A(s)\cap[C(s)\cup A(s)], & s\in(F\backslash G)\cap(H\cap F)=F\cap G'\cap H \\ [A(s)\cup B(s)]\cap C(s), & s\in(F\cap G)\cap(H\backslash F)=\emptyset \\ [A(s)\cup B(s)]\cap[C(s)\cup A(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

$$T(s)=\begin{cases} A(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A(s)\cup B(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cap C(s), & s\in(F\backslash G)\cap(H\backslash F)=\emptyset \\ A(s), & s\in(F\backslash G)\cap(H\cap F)=F\cap G'\cap H \\ [A(s)\cup B(s)]\cap C(s), & s\in(F\cap G)\cap(H\backslash F)=\emptyset \\ [A(s)\cup B(s)]\cap[C(s)\cup A(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\backslash G=F\cap G'$, if $s\in G'$, then $s\in H\backslash G$ or $s\in(G\cup H)'$. Hence, if $s\in F\backslash G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$.

Thus, it is seen that $(N,F)=(T,F)$.

8) $[(A,F)\overset{*}{\underset{*}{\sim}}(B,G)]\tilde{\lambda}(C,H)=[(A,F)\overset{*}{\underset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\overset{*}{\underset{*}{\sim}}(C,H)]$.

Proof: Let $(A,F)\overset{*}{\underset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\lambda}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A'(s)\cup C'(s) & s\in(F\backslash G)\cap H=F\cap G'\cap H \\ [A'(s)\cup B'(s)]\cup C'(s) & s\in(F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\overset{*}{\underset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\overset{*}{\underset{*}{\sim}}(C,H)]$. Let $(A,F)\overset{*}{\underset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A'(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\overset{*}{\underset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$

$$W(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in(F\setminus H)\setminus G=F\cap G'\cap H' \\ A'(s)\cup C'(s), & s\in(F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cup B'(s), & s\in(F\setminus H)\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cup[B'(s)\cup C'(s)], & s\in(F\setminus H)\cap(G\cap H)=\emptyset \\ [A'(s)\cup C'(s)]\cup B'(s), & s\in(F\cap H)\cap(G\setminus H)=\emptyset \\ [A'(s)\cup C'(s)]\cup[B'(s)\cup C'(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that (N,F)=(T,F).

9) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\tilde{\theta}(C,H)=[(A,F)\underset{\backslash}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\tilde{\setminus}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\theta}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M'(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cap C'(s) & s\in(F\setminus G)\cap H=F\cap G'\cap H \\ [A(s)\cap B(s)]\cap C'(s) & s\in(F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\backslash}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\tilde{\setminus}(C,H)]$. Let $(A,F)\underset{\backslash}{\overset{*}{\sim}}(C,H)=(V,F)$, where

$\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\tilde{\setminus}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B(s), & s\in G\backslash H \\ B(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Therefore,

$$T(s)=\begin{cases} A'(s), & s\in (F\backslash H)\backslash G=F\cap G'\cap H' \\ A(s)\cap C'(s), & s\in (F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cap B(s), & s\in (F\backslash H)\cap (G\backslash H)=F\cap G\cap H' \\ A'(s)\cap [B(s)\cap C'(s)], & s\in (F\backslash H)\cap (G\cap H)=\emptyset \\ [A(s)\cap C'(s)]\cap B(s), & s\in (F\cap H)\cap (G\backslash H)=\emptyset \\ [A(s)\cap C'(s)]\cap [B(s)\cap C'(s)], & s\in (F\cap H)\cap (G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

10) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\underset{*}{\sim}(C,H)=[(A,F)\underset{\lambda}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\tilde{\lambda}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\underset{*}{\sim}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M'(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in (F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cup C'(s) & s\in (F\backslash G)\cap H=F\cap G'\cap H \\ [A(s)\cap B(s)]\cup C'(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\lambda}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\tilde{\lambda}(C,H)]$. Let $(A,F)\underset{\lambda}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\tilde{\lambda}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B(s), & s\in G\backslash H \\ B(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}\,(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in (F\backslash H)\backslash G=F\cap G'\cap H' \\ A(s)\cup C'(s), & s\in (F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cap B(s), & s\in (F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cap[B(s)\cup C'(s)], & s\in (F\backslash H)\cap(G\cap H)=\emptyset \\ [A(s)\cup C'(s)]\cap B(s), & s\in (F\cap H)\cap(G\backslash H)=\emptyset \\ [A(s)\cup C'(s)]\cap[B(s)\cup C'(s)], & s\in (F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

11) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\tilde{\cap}(C,H)=[(A,F)\underset{\gamma}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{\gamma}{\overset{*}{\sim}}(C,H)]$

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\cap}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M(s)\cap C(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A'(s)\cap C(s) & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A'(s)\cup B'(s)]\cap C(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\gamma}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{\gamma}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\gamma}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A'(s)\cap C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\gamma}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cap C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}\,(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap H' \\ A'(s)\cap C(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cup B'(s), & s\in (F\setminus H)\cap (G\setminus H)=F\cap G\cap H' \\ A'(s)\cup [B'(s)\cap C(s)], & s\in (F\setminus H)\cap (G\cap H)=\emptyset \\ [A'(s)\cap C(s)]\cup B'(s), & s\in (F\cap H)\cap (G\setminus H)=\emptyset \\ [A'(s)\cap C(s)]\cup [B'(s)\cap C(s)], & s\in (F\cap H)\cap (G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

12) $[(A,F)\underset{*}{\overset{*}{\sim}}(B,G)]\tilde{\cup}(C,H)=[(A,F)\underset{+}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{+}{\overset{*}{\sim}}(C,H)]$

Proof: Let $(A,F)\underset{*}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cup B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\cup}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M(s)\cup C(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A'(s)\cup C(s), & s\in(F\backslash G)\cap H=F\cap G'\cap H \\ [A'(s)\cup B'(s)]\cup C(s), & s\in(F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{+}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{+}{\overset{*}{\sim}}(C,H)]$.Let $(A,F)\underset{+}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A'(s)\cup C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{+}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cup C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}\,(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$
T(s)=\begin{cases}
A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\
A'(s)\cup C(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\
A'(s)\cup B'(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\
A'(s)\cup[B'(s)\cup C(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\
[A'(s)\cup C(s)]\cup B'(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\
[A'(s)\cup C(s)]\cup [B'(s)\cup C(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H
\end{cases}
$$

It is seen that $(N,F)=(T,F)$.

3.1.2. Distribution of soft binary piecewise operations over complementary soft binary piecewise theta operation:

1) $(A,F)\tilde{\mp}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]=[(A,F)\underset{*}{\sim}(B,G)]\tilde{\cap}[(C,H)\underset{*}{\sim}(A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$
M(s)=\begin{cases}
B'(s), & s\in G\backslash H \\
B'(s)\cap C'(s), & s\in G\cap H
\end{cases}
$$

Let $(A,F)\tilde{\mp}(M,G)=(N,F)$, where $\forall s\in F$;

$$
N(s)=\begin{cases}
A(s), & s\in F\backslash G \\
A'(s)\cup M(s), & s\in F\cap G
\end{cases}
$$

Thus,

$$
N(s)=\begin{cases}
A(s), & s\in F\backslash G \\
A'(s)\cup B'(s), & s\in F\cap(G\backslash H)=F\cap G\cap H' \\
A'(s)\cup[(B'(s)\cap C'(s)], & s\in F\cap(G\cap H)=F\cap G\cap H
\end{cases}
$$

Now let's handle $[(A,F)\underset{*}{\sim}(B,G)]\tilde{\cap}[(C,H)\underset{*}{\sim}(A,F)]$. Let $(A,F)\underset{*}{\sim}(B,G)=(V,F)$, where $\forall s\in F$;

$$
V(s)=\begin{cases}
A(s), & s\in F\backslash G \\
A'(s)\cup B'(s), & s\in F\cap G
\end{cases}
$$

Suppose that $(C,H)\underset{*}{\sim}(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\backslash F \\ C'(s)\cup A'(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in (F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cup B'(s), & s\in (F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cap C(s), & s\in (F\backslash G)\cap (H\backslash F)=\emptyset \\ A(s)\cap [C'(s)\cup A'(s)], & s\in (F\backslash G)\cap (H\cap F)=F\cap G'\cap H \\ [A'(s)\cup B'(s)]\cap C(s), & s\in (F\cap G)\cap (H\backslash F)=\emptyset \\ [A'(s)\cup B'(s)]\cap [C'(s)\cup A'(s)], & s\in (F\cap G)\cap (H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\backslash G=F\cap G'$, if $s\in G'$, then $s\in H\backslash G$ or $s\in (G\cup H)'$. Hence, if $s\in F\backslash G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$. Thus, it is seen that $(N,F)=(T,F)$.

2) $[(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)]\tilde{\mp}(C,H)=[(A,F)\underset{\cup}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\cup}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\mp}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M'(s)\cup C(s), & s\in F\cap H \end{cases}$$

$$N(s)=\begin{cases} A'(s), & s\in (F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in (F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cup C(s) & s\in (F\backslash G)\cap H=F\cap G'\cap H \\ [A(s)\cup B(s)]\cup C(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\cup}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\cup}(C,H)]$. Let $(A,F)\underset{\cup}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A(s)\cup C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\tilde{\cup}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B(s), & s\in G\setminus H \\ B(s)\cup C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Hence,

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap H' \\ A(s)\cup C(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cup B(s), & s\in (F\setminus H)\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cup[B(s)\cup C(s)], & s\in (F\setminus H)\cap(G\cap H)=\emptyset \\ [A(s)\cup C(s)]\cup B(s), & s\in (F\cap H)\cap(G\setminus H)=\emptyset \\ [A(s)\cup C(s)]\cup [B(s)\cup C(s)], & s\in (F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

3) $(A,F)\tilde{\gamma}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]=[(A,F)\tilde{}_{\theta}(B,G)]\tilde{\cap}[(C,H)\tilde{}_{\theta}(A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{}_{\gamma}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A'(s)\cap M(s), & s\in F\cap G \end{cases}$$

Thus,

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cap[(B'(s)\cap C'(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\theta}{\sim}(B,G)]\tilde{\cap}[(C,H)\underset{\theta}{\sim}(A,F)]$. Let $(A,F)\underset{\theta}{\sim}(B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\underset{\theta}{\sim}(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\setminus F \\ C'(s)\cap A'(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$ $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cap C(s), & s\in(F\setminus G)\cap(H\setminus F)=\emptyset \\ A(s)\cap[C'(s)\cap A'(s)], & s\in(F\setminus G)\cap(H\cap F)=F\cap G'\cap H \\ [A'(s)\cap B'(s)]\cap C(s), & s\in(F\cap G)\cap(H\setminus F)=\emptyset \\ [A'(s)\cap B'(s)]\cap[C'(s)\cap A'(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\setminus G=F\cap G'$, if $s\in G'$, then $s\in H\setminus G$ or $s\in(G\cup H)'$. Hence, if $s\in F\setminus G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$. Thus, it is seen that $(N,F)=(T,F)$.

4) $[(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)]\tilde{\gamma}(C,H)=[(A,F)\overset{*}{\underset{\cap}{\sim}}(C,H)]\tilde{\cup}[(B,G)\overset{*}{\underset{\cap}{\sim}}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\theta}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M'(s)\cap C(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cap C(s) & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A(s)\cup B(s)]\cap C(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\cap}{\overset{*}{\sim}}(C,H)]\tilde{\cup}[(B,G)\underset{\cap}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\cap}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A(s)\cap C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\cap}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B(s)\cap C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Hence,

$$T(s)=\begin{cases} A'(s), & s\in(F\setminus H)\setminus G=F\cap G'\cap H' \\ A(s)\cap C(s), & s\in(F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cup B'(s), & s\in(F\setminus H)\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cup[B(s)\cap C(s)], & s\in(F\setminus H)\cap(G\cap H)=\emptyset \\ [A(s)\cap C(s)]\cup B'(s), & s\in(F\cap H)\cap(G\setminus H)=\emptyset \\ [A(s)\cap C(s)]\cup [B(s)\cap C(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that (N,F)=(T,F).

5) $(A,F)\tilde{\setminus}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]=[(A,F)\tilde{\cap}(B,G)]\tilde{\cup}[(C,H)\tilde{\cap}(A,F)]$.

Proof: Let $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{\setminus}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cap M'(s), & s\in F\cap G \end{cases}$$

Thus,

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cap B(s), & s\in F\cap(G\setminus H)=F\cap G\cap H' \\ A(s)\cap[(B(s)\cup C(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\tilde{\cap}(B,G)]\tilde{\cup}[(C,H)\tilde{\cap}(A,F)]$. Let $(A,F)\tilde{\cap}((B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cap B(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\tilde{\cap}(A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\setminus F \\ C(s)\cap A(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$ $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A(s)\cap B(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cup C(s), & s\in(F\setminus G)\cap(H\setminus F)=\emptyset \\ A(s)\cup[C(s)\cap A(s)], & s\in(F\setminus G)\cap(H\cap F)=F\cap G'\cap H \\ [A(s)\cap B(s)]\cup C(s), & s\in(F\cap G)\cap(H\setminus F)=\emptyset \\ [A(s)\cap B(s)]\cup[C(s)\cap A(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

$$T(s)=\begin{cases} A(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A(s)\cap B(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cup C(s), & s\in(F\setminus G)\cap(H\setminus F)=\emptyset \\ A(s) & s\in(F\setminus G)\cap(H\cap F)=F\cap G'\cap H \\ [A(s)\cap B(s)]\cup C(s), & s\in(F\cap G)\cap(H\setminus F)=\emptyset \\ [A(s)\cap B(s)]\cup[C(s)\cap A(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\setminus G=F\cap G'$, if $s\in G'$, then $s\in H\setminus G$ or $s\in(G\cup H)'$. Hence, if $s\in F\setminus G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$. Thus, it is seen that $(N,F)=(T,F)$.

6) $[(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)]\tilde{\setminus}(C,H)=[(A,F)\overset{*}{\underset{\theta}{\sim}}(C,H)]\tilde{\cap}[(B,G)\overset{*}{\underset{\theta}{\sim}}(C,H)]$.

Proof: Let $(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\setminus}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A'(s)\cap C'(s), & s\in(F\setminus G)\cap H=F\cap G'\cap H \end{cases}$$

$[A'(s)\cap B'(s)]\cap C'(s), \quad s\in(F\cap G)\cap H=F\cap G\cap H$

Now let's handle $[(A,F)\underset{\theta}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A'(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}\,(W,H)=(T,F)$, where $\forall\, s\in F$

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\ A'(s)\cap C'(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cap B'(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cap[B'(s)\cap C'(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\ [A'(s)\cap C'(s)]\cap B'(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\ [A'(s)\cap C'(s)]\cap[B'(s)\cap C'(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

7) $(A,F)\tilde{\lambda}[(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)]=[(A,F)\tilde{\cup}(B,G)]\tilde{\cup}[(C,H)\tilde{\cup}A,F)]$, where $F\cap G'\cap H=\emptyset$.

Proof: Let $(B,G)\underset{\theta}{\overset{*}{\sim}}(C,H)=(M,G)$, where $\forall s\in G$;

$$M(s)=\begin{cases} B'(s), & s\in G\backslash H \\ B'(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(A,F)\tilde{\lambda}(M,G)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cup M'(s), & s\in F\cap G \end{cases}$$

$$N(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cup B(s), & s\in F\cap(G\setminus H)=F\cap G\cap H' \\ A(s)\cup[(G(s)\cup C(s)], & s\in F\cap(G\cap H)=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\,\tilde{\cup}(B,G)]\tilde{\cup}[(C,H)\tilde{\cup}(A,F)]$. Let $(A,F)\tilde{\cup}(B,G)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A(s), & s\in F\setminus G \\ A(s)\cup B(s), & s\in F\cap G \end{cases}$$

Suppose that $(C,H)\tilde{\cup}A,F)=(W,H)$, where $\forall s\in H$;

$$W(s)=\begin{cases} C(s), & s\in H\setminus F \\ C(s)\cup A(s), & s\in H\cap F \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A(s), & s\in(F\setminus G)\setminus H=F\cap G'\cap H' \\ A(s)\cup B(s), & s\in(F\cap G)\setminus H=F\cap G\cap H' \\ A(s)\cup C(s), & s\in(F\setminus G)\cap(H\setminus F)=\emptyset \\ A(s)\cup[C(s)\cup A(s)], & s\in(F\setminus G)\cap(H\cap F)=F\cap G'\cap H \\ [A(s)\cup B(s)]\cup C(s), & s\in(F\cap G)\cap(H\setminus F)=\emptyset \\ [A(s)\cup B(s)]\cup[C(s)\cup A(s)], & s\in(F\cap G)\cap(H\cap F)=F\cap G\cap H \end{cases}$$

Since $F\setminus G=F\cap G'$, if $s\in G'$, then $s\in H\setminus G$ or $s\in(G\cup H)'$. Hence, if $s\in F\setminus G$, $s\in F\cap G'\cap H'$ or $s\in F\cap G'\cap H$. Thus, it is seen that $(N,F)=(T,F)$.

8) $[(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)]\tilde{\lambda}(C,H)=[(A,F)\underset{*}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{*}{\overset{*}{\sim}}(C,H)]$.

Proof: Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\lambda}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M(s)\cup C'(s), & s\in F\cap H \end{cases}$$

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A'(s)\cup C'(s) & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A'(s)\cap B'(s)]\cup C'(s) & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{*}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{*}{\overset{*}{\sim}}(C,H)]$.Let $(A,F)\underset{*}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A'(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{*}{\overset{*}{\sim}}(C,H)=(W,B)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}\,(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap H' \\ A'(s)\cup C'(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cap B'(s), & s\in (F\setminus H)\cap (G\setminus H)=F\cap G\cap H' \\ A'(s)\cap [B'(s)\cup C'(s)], & s\in (F\setminus H)\cap (G\cap H)=\emptyset \\ [A'(s)\cup C'(s)]\cap B'(s), & s\in (F\cap H)\cap (G\setminus H)=\emptyset \\ [A'(s)\cup C'(s)]\cap [B'(s)\cup C'(s)], & s\in (F\cap H)\cap (G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

9) $[(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)]\tilde{\theta}(C,H)=[(A,F)\overset{*}{\underset{\backslash}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\backslash}(C,H)]$, where $F\cap G\cap H'=\emptyset$

Proof: Let $(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\theta}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M'(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in (F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in (F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cap C'(s), & s\in (F\backslash G)\cap H=F\cap G'\cap H \\ [A(s)\cup B(s)]\cap C'(s), & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\overset{*}{\underset{\backslash}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\backslash}(C,H)]$. Let $(A,F)\overset{*}{\underset{\backslash}{\sim}}(C,H)=(V,F)$, where

$\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A(s)\cap C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\tilde{\backslash}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B(s), & s\in G\backslash H \\ B(s)\cap C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Thus,

$$T(s)=\begin{cases} A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\ A(s)\cap C'(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cup B(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cup[B(s)\cap C'(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\ [A(s)\cap C'(s)]\cup B(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\ [A(s)\cap C'(s)]\cup [B(s)\cap C'(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that (N,F)=(T,F).

10) $[(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)]\underset{*}{\sim}(C,H)=[(A,F)\overset{*}{\underset{\lambda}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\lambda}(C,H)]$, where $F\cap G\cap H'=\emptyset$.

Proof: Let $(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\underset{*}{\sim}(C,H)=(N,F)$, where $\forall s\in F$,

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M'(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A(s)\cup C'(s), & s\in(F\backslash G)\cap H=F\cap G'\cap H \\ [A(s)\cup B(s)]\cup C'(s), & s\in(F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let $[(A,F)\overset{*}{\underset{\lambda}{\sim}}(C,H)]\tilde{\cup}[(B,G)\tilde{\lambda}(C,H)]$. Let $(A,F)\overset{*}{\underset{\lambda}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\backslash H \\ A(s)\cup C'(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\tilde{\lambda}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B(s), & s\in G\backslash H \\ B(s)\cup C'(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\widetilde{\cup}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\backslash H \\ V(s)\cup W(s), & s\in F\cap H \end{cases}$$

Hence,

$$T(s)=\begin{cases} A'(s), & s\in(F\backslash H)\backslash G=F\cap G'\cap H' \\ A(s)\cup C'(s), & s\in(F\cap H)\backslash G=F\cap G'\cap H \\ A'(s)\cup B(s), & s\in(F\backslash H)\cap(G\backslash H)=F\cap G\cap H' \\ A'(s)\cup[B(s)\cup C'(s)], & s\in(F\backslash H)\cap(G\cap H)=\emptyset \\ [A(s)\cup C'(s)]\cup B(s), & s\in(F\cap H)\cap(G\backslash H)=\emptyset \\ [A(s)\cup C'(s)]\cup[B(s)\cup C'(s)], & s\in(F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

11) $[(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)]\widetilde{\cap}(C,H)=[(A,F)\overset{*}{\underset{\gamma}{\sim}}(C,H)]\widetilde{\cap}[(B,G)\overset{*}{\underset{\gamma}{\sim}}(C,H)]$.

Proof: Let $(A,F)\overset{*}{\underset{\theta}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\backslash G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\widetilde{\cap}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\backslash H \\ M(s)\cap C(s), & s\in F\cap H \end{cases}$$

Thus,

$$N(s)=\begin{cases} A'(s), & s\in(F\backslash G)\backslash H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in(F\cap G)\backslash H=F\cap G\cap H' \\ A'(s)\cap C(s) & s\in(F\backslash G)\cap H=F\cap G'\cap H \\ [A'(s)\cap B'(s)]\cap C(s) & s\in(F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{\gamma}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{\gamma}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{\gamma}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A'(s)\cap C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{\gamma}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cap C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

Hence,

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap G' \\ A'(s)\cap C(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cap B'(s), & s\in (F\setminus H)\cap(G\setminus H)=F\cap G\cap H' \\ A'(s)\cap[B'(s)\cap C(s)], & s\in (F\setminus H)\cap(G\cap H)=\emptyset \\ [A'(s)\cap C(s)]\cap B'(s), & s\in (F\cap H)\cap(G\setminus H)=\emptyset \\ [A'(s)\cap C(s)]\cap[B'(s)\cap C(s)], & s\in (F\cap H)\cap(G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

12) $[(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)]\tilde{\cup}(C,H)=[(A,F)\underset{+}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{+}{\overset{*}{\sim}}(C,H)]$.

Proof: Let $(A,F)\underset{\theta}{\overset{*}{\sim}}(B,G)=(M,F)$, where $\forall s\in F$;

$$M(s)=\begin{cases} A'(s), & s\in F\setminus G \\ A'(s)\cap B'(s), & s\in F\cap G \end{cases}$$

Let $(M,F)\tilde{\cup}(C,H)=(N,F)$, where $\forall s\in F$;

$$N(s)=\begin{cases} M(s), & s\in F\setminus H \\ M(s)\cup C(s), & s\in F\cap H \end{cases}$$

$$N(s)=\begin{cases} A'(s), & s\in (F\setminus G)\setminus H=F\cap G'\cap H' \\ A'(s)\cap B'(s), & s\in (F\cap G)\setminus H=F\cap G\cap H' \\ A'(s)\cup C(s), & s\in (F\setminus G)\cap H=F\cap G'\cap H \\ [A'(s)\cap B'(s)]\cup C(s), & s\in (F\cap G)\cap H=F\cap G\cap H \end{cases}$$

Now let's handle $[(A,F)\underset{+}{\overset{*}{\sim}}(C,H)]\tilde{\cap}[(B,G)\underset{+}{\overset{*}{\sim}}(C,H)]$. Let $(A,F)\underset{+}{\overset{*}{\sim}}(C,H)=(V,F)$, where $\forall s\in F$;

$$V(s)=\begin{cases} A'(s), & s\in F\setminus H \\ A'(s)\cup C(s), & s\in F\cap H \end{cases}$$

Suppose that $(B,G)\underset{+}{\overset{*}{\sim}}(C,H)=(W,G)$, where $\forall s\in G$;

$$W(s)=\begin{cases} B'(s), & s\in G\setminus H \\ B'(s)\cup C(s), & s\in G\cap H \end{cases}$$

Let $(V,F)\tilde{\cap}(W,H)=(T,F)$, where $\forall s\in F$;

$$T(s)=\begin{cases} V(s), & s\in F\setminus H \\ V(s)\cap W(s), & s\in F\cap H \end{cases}$$

$$T(s)=\begin{cases} A'(s), & s\in (F\setminus H)\setminus G=F\cap G'\cap H' \\ A'(s)\cup C(s), & s\in (F\cap H)\setminus G=F\cap G'\cap H \\ A'(s)\cap B'(s), & s\in (F\setminus H)\cap (G\setminus H)=F\cap G\cap H' \\ A'(s)\cap [B'(s)\cup C(s)], & s\in (F\setminus H)\cap (G\cap H)=\emptyset \\ [A'(s)\cup C(s)]\cap B'(s), & s\in (F\cap H)\cap (G\setminus H)=\emptyset \\ [A'(s)\cup C(s)]\cap [B'(s)\cup C(s)], & s\in (F\cap H)\cap (G\cap H)=F\cap G\cap H \end{cases}$$

It is seen that $(N,F)=(T,F)$.

4. Conclusion

In this paper, we explore more about soft binary piecewise theta and star operation by examining the relationships between this soft set operation and other types of soft set operations. In this paper, it is aimed to contribute to the soft set literature by obtaining the distributions of soft binary piecewise operations over soft binary piecewise theta and star operations with complement. This paper is a theoretical study of soft sets and some future studies may continue by defining and examining some other distribution rules. For future studies, this research is to serve as a basis for many applications, especially decision making cryptography. Since soft set is a powerful mathematical tool for uncertain object detection, with this study, researchers may suggest some new encryption based on soft sets.

Author’s Contribution

The contribution of authors is equal.

Conflict of Interest

The authors have declared that there is no conflict of interest.

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