Kadirli Uygulamalı Bilimler Fakültesi Dergisi
Cilt 2, Sayı 2, 270-280, 2022

Journal of Kadirli Faculty of Applied Sciences
Volume 2, Issue 2, 270-280, 2022

Kadirli Uygulamalı Bilimler Fakültesi Dergisi

Journal of Kadirli Faculty of Applied Sciences

Nonlocal Transformations of Force-Free Duffing-van der Pol equation

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Research Article

Article History:
Received: 10.11.2022
Accepted: 26.11.2022
Available online: 09.12.2022

Keywords:
Nonlocal transformations
Sundman transformation
First integral

ABSTRACT

In this study, the nonlocal transformation methods applied to the nonlinear differential equations and the first integrals obtained by using these transformations are examined. It is shown that the linearized equations by these nonlocal transformations can be integrated by the first integrals. Then, the force-free Duffing-van der Pol oscillator equation is considered, and it is demonstrated this equation with a specific nonlinear term is integrable. To do them, first, this equation is classified by using special functions. Then, an effective procedure is emphasized to obtain a nonlocal transformation pair called Sundman. The Sundman transformation pair is found by concerning this classification. The first integrals of this equation are acquired by this Sundman transformation pair.

Force-Free Duffing Van der Pol Denkleminin Yerel Olmayan Dönüşümleri

Araştırma Makalesi

Makale Tarihçesi:
Geliş tarihi: 10.11.2022
Kabul tarihi:26.11.2022
Online Yayınlanma: 09.12.2022

Anahtar Kelimeler
Yerel olmayan dönüşümler
Sundman dönüşümü
İlk integral

ÖZ

Bu çalışmada, lineer olmayan diferansiyel denklemlere uygulanan yerel olmayan dönüşüm yöntemleri ve bu dönüşümler kullanılarak elde edilen ilk integraller incelenmiştir. Bu yerel olmayan dönüşümler ile lineerleştirilen denklemlerin ilk integraller yardımıyla integre edilebileceği gösterilmiştir. Daha sonra, Duffing-van der Pol denklemi ele alınmış ve lineer olmayan özel bir terime sahip olan bu denklemin integrallenebilir olduğu kanıtlanmıştır. Bu işlemleri yapabilmek için önce bu denklem özel fonksiyonlar kullanılarak sınıflandırılmış, sonra Sundman adı verilen yerel olmayan bir dönüşüm çifti elde etmek için etkili bir yaklaşım açıklanarak, bu sınıflandırmaya karşılık gelen Sundman dönüşüm çifti hesaplanmıştır. Son olarak, bu denklemin ilk integralleri, elde edilen Sundman dönüşüm çifti kullanılarak bulunmuştur.

To Cite: Orhan Ö., 2022. Nonlocal transformations of force-free Duffing-van der pol equation. Kadirli Uygulamalı Bilimler Fakültesi Dergisi, 2(2): 270-280.

Introduction

Many researchers have a significant interest in linearization through a transformation involving nonlocal terms in recent years (Duarte, Moreira and Santos, 1994; Chandrasekar, Senthilvelan and Lakshmanan, 2005). One of the nonlocal transformations is given as

$$X = F(t, x), \qquad dT = G(t, x)dt, \tag{1}$$

which is called the generalized Sundman transformation (Euler 2003; Euler 2004). Here F and G are arbitrary smooth functions. This transformation is called S-transformation, and the equations that can be linearized by using S-transformation are called S-linearizable (Muriel and Romero, 2009). Duarte et al. (Duarte, Moreira and Santos, 1994) show that S-linearizable equations should be in the following form

$$\ddot{x}+ a_2(t,x)\, \dot{x}^2+a_1(t,x)\, \dot{x}+a_0(t,x)=0. \tag{2}$$

Nonlinear equations can be transformed into S-linearizable with these nonlocal transformations, and then, S-transformation pairs of S-linearizable equations can be found. Moreover, the first integrals of the equations can be obtained by using these S-transformation pairs.

The first integrals and solutions of nonlinear equations are interesting in enormous attention in the literature since these equations are essential in applied mathematics, physics, and engineering problems (Orhan and Özer, 2016). Sundman transformations have different generalizations in the literature (Chandrasekar, Senthilvelan and Lakshmanan, 2006). In addition, using generalized Sundman transformation (1), Sundman symmetries can be obtained (Euler and Euler, 2004).

In order to obtain first integrals, different methods are introduced by many authors, and are given as follows; Noether theorem (Noether, 1971), linearization methods (Duarte, Moreira and Santos, 1994; Chandrasekar, Senthilvelan and Lakshmanan, 2005), variational derivatives (Ibragimov, 2006), Lie symmetries (Ashyralyev, Dal and Pınar, 2011; Kopçasız and Yaşar, 2022) and symmetry methods (Orhan and Özer, 2016). One of these feasible methods to derive the first integrals is obtaining transformation pairs, and many methods are defined to find transformation pairs.

There are two types of differential equations the linear and nonlinear differential equations. Finding exact solutions to linear equations can be easier than finding solutions to nonlinear differential equations. Moreover, finding solutions for nonlinear differential equations is more complex than obtaining numerical solutions to nonlinear differential equations and analytic solutions to linear equations. Therefore, getting exact solutions is easier if the nonlinear differential equations can be converted to linear ones. The nonlinear differential equations can be transformed into linear equations by using transformation pair, so they could be linearized.

Our aim in this research, we investigate the first integrals of the force-free Duffing-van der Pol equation by applying the generalized Sundman transformation method to this equation. To construct them, firstly, we will find the Sundman pair by applying the necessary procedure according to the classified equation and appropriate classification. Then, the first integrals are obtained by using this pair of transformation. The studies in literature show that the general form of the first integral for the force-free Duffing-van der Pol equation, which we discuss in this study, is not obtained. Since this equation contains the cubic nonlinear term, which means that its exact solutions could not be found, so the study about the absence of these solutions is given as follows (Panayotounakos et al., 2003).

Duffing-van der Pol equation does not have the first integral containing arbitrary functions and analytical solutions. Chandarasker et al. obtained the first integral by considering with special choices for $\alpha = 4/\beta$ and $\alpha = -3/\beta^2$ for the arbitrary functions which are accounted in the equation.

In this study, we obtain the first integrals for the general form of these types of functions without making any special selections for arbitrary functions. The first integral for the general form of this equation has not been obtained yet. If one more transformation is performed with the help of the first integral obtained, analytical solutions for the general form of the equation can also be found.

The Method for Constructing Transformation Pairs

In this section, we investigate the S-transformation pairs and S-linearizable equations. It is known that these equations have first integral in form

$$\mathrm{A}(\mathrm{t},\mathrm{x})\dot{x} + \mathrm{B}(\mathrm{t},\mathrm{x}). \tag{3}$$

Equation (2) is linearized by using these first integrals, and to perform it; we classify equations to derive the first integrals in this form. To classify the equation, these functions are defined as

$$S_1(t,x) = a_{1x} - 2a_{2t}, \tag{4}$$

$$S_2(t,x) = (a_0 a_{2+} a_{0x})_x + (a_{2t-} a_{1x})_t + (a_{2t-} a_{1x})a_1. \tag{5}$$

After these definitions, the function $S_1(t,x)$ is computed; if $S_1 = 0$, then the function S_2 should be zero. If the function $S_1 \neq 0$, then two different functions should be used. These can be given

$$S_3(t,x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t-}a_{1x}), \qquad (6)$$

$$S_4(t,x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1 \left(\frac{S_2}{S_1}\right) + a_0 a_{2+} a_{0x} \qquad (7)$$

If the function S_3 is computed as zero for these two new functions, then it is seen that $S_4 = 0$. Two different linearizing procedures are used with respect to this classification, and the appropriate procedure is chosen for the considered equation according to obtaining classification results. We investigate the following propositions to explain these procedures that give first integrals by nonlocal transformations (Muriel and Romero, 2010).

Theorem 1: The equation (2) has Sundman transformation pair if and only if it has a first integral as $\mathrm{A(t,x)}\dot{x} + \mathrm{B(t,x)}$.

If an Sundman transformation pair is known then a first integral can be determined of (2). On the other hand, if a first integral of (2) is known, then an Sundman transformation pair can be constructed.

Theorem 2: We take equation (2) and S_1 is calculated. The analysis of these functions leads us to consider two cases:

Case I: We suppose that $S_1 = 0$ and then equation (2) has transformation pair (1) if $S_2 = 0$.

Case II: Let $S_1 \neq 0$; in this situation equation (2) has transformation pair (1) if $S_3 = 0$ and $S_4 = 0$.

Approaches give Sundman transformation pair under these cases are presented like this:

Case I: If $S_1 = S_2 = 0$.

In order to find transformation pair of (2), firstly the function f is defined by

$$f(t,x) = a_0\, a_{2+}\, a_{0x} - \frac{1}{2} a_{1x} - \frac{1}{4} {a_1}^2. \qquad (8)$$

The function $w(t)$ is a solution of the following equations

$$w_t + w^2 + f = 0, \qquad (9)$$

$$w_x = 0. \qquad (10)$$

The function C is a solution of the system

$$C_t = a_0 - C(t,x)\left(\frac{a_1}{2} + w(t)\right), \tag{11}$$

$$C_x = (a_1/2 - w(t)) - C(t,x)a_2. \tag{12}$$

P is determined by solving following equations

$$P_t = \frac{1}{2}a_1, \quad \text{and} \qquad P_x = a_2. \tag{13}$$

F is derived from

$$F_t = CF_x \tag{14}$$

The function G is yielded by

$$G = F_x\, e^{(-P-\int w(t)dt\,)}. \tag{15}$$

Thus, S-transformation pair F and G are found.

Moreover, the coefficients A and B are computed as

$$A(t,x) = \frac{F_x}{G} \quad \text{and} \qquad B(t,x) = \frac{F_t}{G} \tag{16}$$

Hence, the first integrals of the equation (2) are yielded.

Case II: $S_1 \neq 0$ and $S_3 = S_4 = 0$.

For this case, the function C is obtained by using the following equations

$$C_t = a_0 - C(t,x)(a_1 + u(t,x)), \tag{17}$$

$$C_x = -u(t,x) - C(t,x)a_2, \tag{18}$$

where u is a solution of the following system

$$u_x - (a_{2t} - a_{1x}) = 0, \tag{19}$$

$$u_t + u^2 + a_1\, u + a_0\, a_2 + a_{0x} = 0. \tag{20}$$

Later, we should find the function P and to do it, the derivatives are written as

$$P_t = \frac{S_2}{S_1} + a_1, \quad \text{and} \qquad P_x = a_2. \tag{21}$$

And also F

$$F_t = CF_x\,. \tag{22}$$

The function G is yielded by

$$G = F_x \, e^{(-P\,)} \, . \tag{23}$$

Thus, S-transformation pair F and G are found.

Furthermore, the coefficients A and B are computed as

$$A(t,x) = \frac{F_x}{G} \quad \text{and} \quad B(t,x) = \frac{F_t}{G} \tag{24}$$

Hence, the first integrals of equation (2) are yielded for Case II.

The coefficients of the first integrals are derived for Case II.

Sundman Transformation Pairs of Force-Free Duffing-van der Pol Equation

In this section, we give our attention to force-free Duffing-van der Pol equation

$$\ddot{x} + (\alpha + \beta \, x^2)\dot{x} - \gamma \, x \; + x^3 \; = 0, \tag{25}$$

which is second-order ordinary differential equations of the Painlev´e-Gambier classification. In this equation, x is the position coordinate which is a function of the time t, and γ is a scalar parameter demonstrating the damping's nonlinearity and strength (Van der Pol, 1922). DvdP equation is an autonomous equation expressing voltage pulses' dispersion along a neuronal axon.

DvdP equation has received a lot of attention recently by many researchers because it arises in a model describing the propagation of voltage pulses along a neuronal axon. Generally, non-integrability properties have been discussed because the forced version of equation (25) displays a rich diversity. We have different forms of DvdP equation according to choices of nonlinear terms. If the β is chosen zero, equation (25) becomes the force-free Duffing oscillator whose integrability and non-integrability properties, or if the cubic term is absent it becomes the standard van der Pol equation. Equation (25) is yielded stable oscillations, renamed relaxation oscillations, and its current name is a limit cycle type in electrical circuits using vacuum tubes.

This equation is so famous in many areas, such as physics, biology, sociology and even economics, because this equation has not only physical meaning but also biological meaning. Therefore, it is used to model electrical circuits on the one hand and to measure the electrical potentials of neurons in the stomachs of living things on the other hand.

Moreover, this equation was used to model the action potentials of neurons (Fitzhugh, 1961; Nagumo, 1962). Additionally, it is used in seismology as a model of the two plates in a geological fault and phonation as a model for the right and left vocal fold oscillations (Lucero and Schoentgen, 2013). Thus, earthquake faults with viscous friction can be characterized by this equation.

The analytic solutions of the oscillator equations with nonlinear damping are not still explored by researchers, therefore, articles are mostly interested in damped free oscillator equations (Mendelson, 1970). In addition to this, Panayotounakos and his collobrates demonstrated this equation is not analytic without linear stiffness terms (Panayotounakos et. al, 2002); so, researchers have investigated for approximate solutions to this equation by using numerical methods. The approximate solutions of this equation are obtained by a new homotopy perturbation method, the Runge-Kutta method (Khan et al., 2011), and the differential transform method (Mukherjee et al., 2011). Then, Chandrasekar et al. (Chandrasekar et al., 2004) examined the first integrals and exact solutions of this equation with special choices for $\alpha = 4/\beta$ and $\alpha = -3/\beta^2$. As can be seen from these studies, the first integral was found for some special cases, so the first integrals for the general form of DvdP equation have not been found previously. The first integral for the general form of this equation (25) is obtained in this article by using Sundman transformation pair. The force-free Duffing-van der Pol oscillator equation has the following transformation pair F and G where then this transformation pair can linearize equation (25).

We now apply the procedure examined in the previous section to find the Sundman transformation pair, and the first integral of DvdP equation by using them. It is known that we should classify this equation with respect to given functions to apply the procedure to classify equation (25) by computing the functions $S_1,\ S_2,\ S_3$ and S_4. Here, $S_1 = 2\alpha x$ is found, and it is shown that the function $S_1 \neq 0$. Hence, the functions S_3 and S_4 should be equal to zero. We calculate these functions to complete classification and find $S_3 = 0$ and the function S_4 as

$$S_4 = \frac{9-3\alpha\beta-\beta^2\gamma}{\beta^2}. \tag{26}$$

Since S_4 should be equal to zero, we suppose that

$$9 - 3\alpha\beta - \beta^2\gamma = 0. \tag{27}$$

And from (27), the parameter γ is yielded

$$\beta = \frac{3(-\alpha+\sqrt{\alpha^2+4\gamma})}{2\gamma} \tag{28}$$

We can say that equation (25) is classified, and it is in the second class. Hence, case II should be applied to this equation to obtain Sundman transformation pair and its first integrals. To get them, first we find the function $u(x,t)$ as

$$u(x,t) = \frac{(\alpha^2 + 2\gamma - \alpha\sqrt{\alpha^2 + 4\gamma})(\gamma - 3x^2)}{\gamma(-\alpha + \sqrt{\alpha^2 + 4\gamma})}. \tag{29}$$

Using equations (17), (18), the following system is reached

$$C_t = -\frac{\left(-\alpha + \sqrt{\alpha^2 + 4\,\gamma}\right)x(\gamma - x^2) + 2\,\gamma\,C[t,x]}{-\alpha + \sqrt{\alpha^2 + 4\,\gamma}}, \tag{30}$$

$$C_x = -\frac{\left(\alpha^2 + 2\,\gamma - \alpha\sqrt{\alpha^2 + 4\,\gamma}\right)(\gamma - 3x^2)}{\gamma\left(-\alpha + \sqrt{\alpha^2 + 4\gamma}\right)}. \tag{31}$$

If we solve equations (30), (31), the function C is derived as

$$C(t,x) = -\frac{\left(-\alpha + \sqrt{\alpha^2 + 4\,\gamma}\right)(\gamma\,x - x^3)}{2\,\gamma} + H(t). \tag{32}$$

Thus, the function F is obtained as

$$F(t,x) = \varphi\left[\frac{\sqrt{\alpha^2 + 4\,\gamma}\,t - \mathrm{Log}\left[e^{\alpha\,t} - \frac{\gamma e^{\alpha t}}{x^2}\right]}{2\,\gamma}\right]. \tag{33}$$

We find that the following equations

$$P_x = 0, \tag{34}$$

$$P_t = \frac{2\,\gamma}{-\alpha + \sqrt{\alpha^2 + 4\,\gamma}}. \tag{35}$$

Hence, the function P is constructed

$$P(t) = \frac{2\,\gamma\,t}{-\alpha + \sqrt{\alpha^2 + 4\,\gamma}} \tag{36}$$

Using obtained functions P, F, C, and u, the S-transformation pair can be obtained as

$$G(t,x) == -\frac{e^{\alpha\,t - \frac{2\,\gamma\,t}{-\alpha + \sqrt{\alpha^2 + 4\,\gamma}}}\varphi'\left[\frac{\sqrt{\alpha^2 + 4\gamma}t - \mathrm{Log}[e^{\alpha t} - \frac{\gamma\,e^{\alpha t}}{x^2}]}{2\,\gamma}\right]}{(e^{\alpha t} - \frac{\gamma e^{\alpha t}}{x^2})x^3}. \tag{37}$$

It can be seen that from the Theorem 1, first integrals are constructed by this transformation pair below

$$A(t,x) = \frac{D(F(t,x),x)}{G(t,x)} = e^{\frac{2\gamma t}{-\alpha+\sqrt{\alpha^2+4\gamma}}} \tag{38}$$

$$B(t,x) = \frac{D(F(t,x),t)}{G(t,x)} = \frac{-\alpha+\sqrt{\alpha^2+4\gamma}\, e^{\frac{1}{2}\left(\alpha+\sqrt{\alpha^2+4\gamma}\right)t} x(\gamma-x^2)}{2\gamma} \tag{39}$$

Finally, the first integrals of DvdP equation

$$I = \frac{e^{\frac{1}{2}(\alpha+\sqrt{\alpha^2+4\gamma})t}((-\alpha+\sqrt{\alpha^2+4\gamma})x(2\gamma-x^2)+2\gamma x')}{2\gamma}. \tag{40}$$

is obtained. Thus, the general form of the first integral for the Duffing-van der Pol equation is found by Sundman transformation pair.

Conclusions

In this paper, the force-free Duffing-van der Pol equation (25) is considered with nonlinear damping. This equation is an autonomous equation expressing the dispersion of voltage pulses along a neuronal axon and this problem is highly nonlinear in the sense of a mathematical point of view. Also, Sundman transformation pair of DvdP equation is investigated. Then, the first integrals in the general form of this equation are derived by obtained transformation pair. In order to construct these structures firstly, DvdP equation is characterized, and then the transformation can be applied. The first integrals including arbitrary parameters of the equation are found by these operations. These first integrals can be classified for different choices of arbitrary parameters as well.

Since the analytic solutions of this equation (25) could not be found, its numerical solutions were investigated, and then the first integrals were found for some special cases. First integrals for the general form of this equation have not been introduced previously.

In conclusion, the first integrals which are derived by Sundman transformation pair for the general form of DvdP equation are first obtained in this study. This study is important because the general form of the first integral for the Duffing-van der Pol equation is obtained for the first time in the literature.

Statement of Conflict of Interest

Author has declared no conflict of interest.

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