Kadirli Uygulamalı Bilimler Fakültesi Dergisi
Cilt 5, Sayı 1, 70-105, 2025

Journal of Kadirli Faculty of Applied Sciences
Volume 5, Issue 1, 70-105, 2025

Kadirli Uygulamalı Bilimler Fakültesi Dergisi

Journal of Kadirli Faculty of Applied Sciences

Solution of Initial Boundary Value Problems for the Nonlinear Stationary Quasi-Optical Equation with Special Gradient Term

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Research Article

Article History:
Received: 11.01.2025
Accepted: 04.02.2025
Available online: 17.03.2025

Keywords:
Quantum mechanics
Nonlinear quasi-optical equation
Initial boundary value
Problem
Existence and uniquenes
Gradient term analysis

ABSTRACT

This study examines a fundamental problem in quantum mechanics: the initial boundary value problem for a stationary, nonlinear quasi-optical equation with a specific gradient. This complex equation, which surpasses the previously studied nonstationary linear Schrödinger equation, offers more accurate models of particle behavior at the microscopic level. The research defines the problem as a first-kind problem, then meticulously formulates the necessary and sufficient conditions for its solution. Theorems on the existence and uniqueness of the solution are proven, and an estimate of the solution is obtained. This work is significant for quantum mechanics and optics, particularly in understanding electromagnetic wave propagation when wavelengths are comparable to the optical system's dimensions. The findings contribute to optical design and engineering applications, offering new insights beyond classical physics.

Özel Gradyent Terimli Doğrusal Olmayan Durgun Kuazi-Optik Denklemi İçin Başlangıç Sınır Değer Problemlerinin Çözümü

Araştırma Makalesi

Makale Tarihçesi:
Geliş tarihi: 11.01.2025
Kabul tarihi: 04.02.2025
Online Yayınlanma:17.03.2025

Anahtar Kelimeler:
Kuantum mekaniği
Doğrusal olmayan yarı-optik
Denklem başlangıç sınır değeri
Problem
Varoluş ve benzersizlikler
Gradyan terim analizi

ÖZ

Bu çalışma, kuantum mekaniğindeki temel bir problemi incelemektedir: belirli bir gradyana sahip durağan, doğrusal olmayan yarı-optik bir denklem için başlangıç sınır değer problemi. Daha önce çalışılan durağan olmayan doğrusal Schrödinger denklemini aşan bu karmaşık denklem, mikroskobik düzeyde parçacık davranışının daha doğru modellerini sunmaktadır. Araştırma, problemi birinci türden bir problem olarak tanımlıyor, ardından çözümü için gerekli ve yeterli koşulları titizlikle formüle ediyor. Çözümün varlığı ve tekliği üzerine teoremler kanıtlanıyor ve çözümün bir tahmini elde ediliyor. Bu çalışma kuantum mekaniği ve optik için, özellikle de dalga boyları optik sistemin boyutlarıyla karşılaştırılabilir olduğunda elektromanyetik dalga yayılımını anlamak açısından önemlidir. Bulgular, klasik fiziğin ötesinde yeni anlayışlar sunarak optik tasarım ve mühendislik uygulamalarına katkıda bulunmaktadır.

To Cite: Yagub G, Vural M., 2025. Solution of initial boundary value problems for the nonlinear stationary quasi-optical equation with special gradient term. Kadirli Uygulamalı Bilimler Fakültesi Dergisi, 5(1): 70-105.

Introduction

Initial boundary value problems for linear and nonlinear stationary quasi-optical or non-stationary Schrödinger equations often arise in quantum mechanics, nuclear physics, nonlinear optics and various contemporary areas of physics and technology, and the study of these problems is of both theoretical and practical importance (Butkovsky, 1973; Zhuravle, 2001). One such problem is the problem of the motion of charged particles. As is well known, if charged particles move in a constant homogeneous magnetic field and the direction of the magnetic field is along the z-axis, then the motion of such particles takes place in the (x, y) $\in E_2$ plane, and this motion is usually described by the two-dimensional linear non-stationary Schrödinger equation with a special gradient term (see (Butkovsky, 1973) page 182, (Zhuravlev, 2001)).

Initial boundary value problems for linear and nonlinear quasi-stationary quasi-optical or non-stationary Schrödinger equations without special gradient term were first considered in (Iskenderov and Yagubov, 1988; Iskenderov and Yagubov, 2012) and initial boundary value problems for linear and non-linear quasi-optical or non-stationary Schrödinger equations with special gradient term were first considered in (Toyaoğlu, 2012; Zengin, 2021). It should be noted that the initial boundary value problems for the non-stationary linear Schrödinger equation without special gradient term or for the stationary linear-quasi, quasi-optical equation without special gradient term in the case where the coefficients of the equation are measurable bounded functions depending only on the time variable, but not necessarily differentiable with respect to the time variable, have been previously studied in (Iskenderov and Yagubov, 2012). It should be noted that the non-stationary linear Schrödinger equation with special gradient term and complex potential or the second kind of initial boundary value problem for the linear stationary quasi-optical equation with a special gradient term was first considered in (Yagubov et al., 2022). However, the initial boundary value problems for the nonlinear stationary quasi-optical equation with a special gradient term have not been investigated when the refraction and absorption coefficients of the nonlinear stationary quasi-optical equation are measurable bounded functions depending only on its variable. Therefore, the topic of this study on the initial boundary value problems for the nonlinear stationary quasi-optical equation with special gradient term is topical and of theoretical and practical importance.

The Statement of the Initial Boundary Value Problem of First Kind

Let B be a Banach space and $0 < l, 0 < L$ be given real numbers. $C^k([0, L], B)$ is the Banach space of functions defined on the interval k times continuously differentiable and whose values belong to the Banach space B, $L_p(0, l)$ is the Lebesgue space of functions for which the p-th power of their absolute value is Lebesgue integrable, $L_2(0, L, ; B)$ is the Banach space of square integrable functions defined on the interval $(0, L)$, whose values belong to the Banach space $B, L_\infty(0, L; B)$ is the Banach space of measurable bounded functions defined on the interval $(0, L)$ whose values belong to the Banach space B and $W_p^k(0, l), W_p^{k,m}(\Omega)$ (where $1 \leqslant p, 0 \leqslant k, 0 \leqslant m$), $\Omega = \Omega_t = (0, l) \times (0, t))$ is the Sobolev spaces and have been defined, for example, in (Ladyzhenskaya and Salonnikov, 1967; Lions, 1967).

Let $0 \leqslant x \leqslant l$ and $0 \leqslant z \leqslant L$. Consider the following initial boundary value problem related to finding the function $\psi = \psi(x, z)$.

$$i\frac{\partial\psi}{\partial z} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1(x,z)\frac{\partial\psi}{\partial x} - a(x)\psi + v_0(z)\psi + i_1(z)\psi + a_2|\psi|^2\psi = f(x,z) \quad (2.1)$$

$$\psi(x, 0) = \varphi(x) \quad (2.2)$$

$$\psi(0, z) = \psi(l, z) = 0 \quad (2.3)$$

where $(x, z) \in \Omega, x \in (0, l), t \in (0, L), 0 < a_0 \in \mathbb{R}, a_2 \in \mathbb{C}$ with

$$a_2 = \mathrm{Re}a_2 + i.\mathrm{Im}a_2, \mathrm{Im}a_2 > 0, \mathrm{Re}a_2 < 0, \mathrm{Im}a_2 \geqslant 2|\mathrm{Re}a_2| \quad (2.4)$$

$a(x), a_1(x, z), v_0(z), v_1(z)$ are real-valued measurable functions and satisfy the following conditions

$$0 \leqslant a(x) \leqslant \mu_1, \left|\frac{da(x)}{dx}\right| \leqslant \mu_2, \left|\frac{d^2a(x)}{dx^2}\right| \leqslant \mu_3 \quad (2.5)$$

$$|a_1(x,z)| \leqslant \mu_4, \left|\frac{\partial a_1(x,z)}{\partial x}\right| \leqslant \mu_5, \left|\frac{\partial^2 a_1(x,z)}{\partial x^2}\right| \leqslant \mu_6 \quad (2.6)$$

$$|v_s(z)| \leqslant b_s \quad (2.7)$$

where $\overset{0}{\forall} x \in (0, l), \mu_1, \mu_2, \mu_3 =$const$> 0, \overset{0}{\forall}(x, z) \in \Omega, a_1(0, z) = a_1(l, z) = 0, z \in (0, T)$ $\mu_3, \mu_4, \mu_5, \mu_6 =$ const $> 0, s = 0,1, \overset{0}{\forall} z \in (0, L), b_0, b_1 =$const$> 0$ $\varphi(x), f(x, z)$ are complex valued measurable functions and satisfy the following conditions

$$\varphi \in \overset{0}{W}{}_2^2(0, l), f \in \overset{0}{W}{}_2^{2,1}(\Omega) \quad (2.8)$$

The solution of the initial boundary value problem $(2.1)-(2.3)$ is a function $\psi \in W_2^{0,1}(\Omega)$ satisfying equation $(2,1)$ $\overset{0}{\forall}(x,z) \in \Omega$, initial value condition $(2,2)$ $\overset{0}{\forall} x \in (0,l)$ and boundary value condition $(2,3)$ $\overset{0}{\forall} z \in (0,L)$.

The Solution of the Initial Boundary Value Problem of First Kind

In this section we will use Galerkin's method to prove the existence and uniqueness theorem for the solution of the initial boundary value problem (2.1) - (2.3).

Theorem 3.1. Suppose that the complex number a_2 and the functions $a(x), a_1(x,z), \varphi(x), f(x,z)$ satisfy the conditions $(2{,}4-2{,}8)$. Then the initial boundary value problem (2.1)-(2.3) has only one almost everywhere solution that belongs to the space $\overset{0}{W}{}_2^{2,1}$. Moreover the following inequality holds.

$$\|\psi\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)} \le c_0\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)} + \|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)} + \|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right) \tag{3.1}$$

where $0 < c_0$ is a constant.

Proof. Consider any system of functions $u_k = u_k(x), k = 1,2, \ldots$ belongs to $\overset{0}{W}{}_2^{2,1}(0,l)$ which is orthonormal in $L_2(0,l)$. Let us take the solutions of the eigenvalue problem

$$\Lambda X(x) = \lambda X(x), x \in (0,l), X(0) = X(l) = 0 \tag{3.2}$$

corresponding to the eigenvalues $\lambda = \lambda_k,\ k = 1{,}2$ of the eigenvalue problem (3,2) as a system of functions. The opertor Λ in (3.2) defined as below,

$$\Lambda = -a_0 \frac{d^2}{dx^2} + a(x) \tag{3.3}$$

It must be said that the eigenvalue problem (3,2) is studied in (Ladyzhenskaya and Salonnikov, 1967) and it has been shown that the problem (3,2) has solutions $X = u_k(x), k = 1,2, \ldots$ for $\lambda = \lambda_k,\ k = 1,2, \ldots$. Moreover, these solutions form a basis for $\overset{0}{W}{}_2^1(0,l), \overset{0}{W}{}_2^2(0,l)$. On the other hand, these solutions satisfy the conditions of orthonormality in $L_2(0,l)$ and orthogonality in $\overset{0}{W}{}_2^1(0,l), \overset{0}{W}{}_2^2(0,l)$. In other words, it provides the following relations,

$$(u_k, u_m)_{L_2(0,l)} = \int_0^l u_k(x) u_m(x) dx = \delta_k^m \tag{3.4}$$

$$[u_k, u_m] = (\Lambda u_k, u_m)_{L_2(0,l)} = \int_0^l \left(a_0 \frac{du_k}{dx} \frac{du_m}{dx} + a(x) u_k u_m \right) dx = \lambda_k \delta_k^m, \tag{3.5}$$

$$\{u_k, u_m\} = (\Lambda u_k, \Lambda u_m)_{L_2(l)} = \lambda_k \delta_k^m \tag{3.6}$$

where $k, m = 1,2 \ldots \delta_k^m$ is the Kronecker constants defined as below

$$\delta_k^m = \begin{cases} 1, k = m \\ 0, k \neq m, k, m = 1,2 \end{cases} \tag{3.7}$$

according to our assumption, since the condition $0 \leqslant a(x)$ is satisfied, the eigenvalues $\lambda = \lambda_k, k = 1,2 \ldots$ are real, positive and satisfy the conditions $\lambda_i \leqslant \lambda_j \ldots$ whenever $i \leqslant j$ and $\lim_{k \to \infty} \lambda_k = +\infty$.

In addition, let us assume that $\|u_k\|_{W_{W_2(0,l)}^2} \leqslant \tilde{d}_k < +\infty$ is finite for any $k = 1,2, \ldots$ holds where $\tilde{d}_k$ *is a positive constants*.

According to the Galerkin's method, we will search for an approximate solution of the initial boundary value problem (2.1)-(2.3) in the following form

$$\psi^N(x, z) = \sum_{k=1}^{N} c_k^N(z) u_k(x) \tag{3.8}$$

the coefficients $c_k^N(t) = (\psi^N(.,t), u_k)_{L_2(0,l)}, k = 1,2, \ldots, N$ are found from the following conditions:

$$i \frac{d}{dz} (\psi^N(.,z), u_k)_{L_2(0,l)} - (\Lambda \psi_N(.,z), u_k)_{L_2(0,l)} + i \left(a_1(.,z) \frac{\partial \psi^N(.,z)}{\partial x, u_k} \right)_{L_2(0,l)}$$
$$+ (a_2 |\psi^N(.,z)|^2 \psi_N(.,z), u_k)_{L_2(0,l)} = f_k(z)$$

where $f_k(z) = (f(.,z), u_k)_{L_2(0,l)}, \varphi_k = (\varphi, u_k)_{L_2(0,l)}, k = 1,2, \ldots, N$.

As can be seen, the system $(3,9)$ is a system of a number of nonlinear first order ordinary differential equations with variable coefficients and right-hand side of each equations is f_k which is in $L_2(0, L)$. As known from the theory of ordinary differential equations (3.9), (3.10) is a Cauchy problem and has at least one solution that belongs to $W_2^1(0,L)$ (Alekseyev, 1979; Vasilyev, 1980).

Lemma 3.2. For the solution of the system (3.9) - (3.10) the following inequality holds

$$\int_0^L \sum_{k=1}^N \left|c_k^N(z)\right|^2 dz + \int_0^L \sum_{k=1}^N \left|\frac{dc_k^N(z)}{dz}\right|^2 dz \le \left\|\psi^N\right\|^2_{\overset{0}{W}{}_2^{2,1}(\Omega)} \le \tag{3.11}$$

$$\le c_0\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(0,l)} + \left\|f\right\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(0,l)} + \left\|f\right\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right),\ N = 1,2,\ldots\quad .$$

Proof. Let us multiply the k^{th} equation of (3,9) by $c_{\bar{k}}^{-N}(z)$, add the obtained equations from 1 to N over k and integrate from 0 to L over $z \leqslant L$. If we apply partial integration to the obtained equation by using the conditions $u_k(0) = u_k(l) = 0$, we find the following equation $\forall z \in [0,L]$.

$$\int_{\Omega_z}\left(i\frac{\partial\psi^N}{\partial z}\bar{\psi}^N - a_0\left|\frac{\partial\psi^N}{\partial x}\right|^2 + ia_1(x,\tau)\frac{\partial\psi^N}{\partial x}\bar{\psi}^N - a(x)|\psi^N|^2\right)dxad\tau +$$
$$+\int_{\Omega_z}(v_0(\tau)|\psi^N|^2 + iv_1(\tau)|\psi^N|^2 + a_2|\psi^N|^4)dxd\tau = \int_{\Omega_z} f(x,\tau)\bar{\psi}^N(x,\tau)dxd\tau$$

if we subtract its complex conjugate from this equation, we obtain the following equation

$$i\int_{\Omega_z}\left(\frac{\partial\psi^N}{\partial z}\bar{\psi}^N + \frac{\partial\bar{\psi}^N}{\partial z}\psi^N\right)dxd\tau + i\int_{\Omega_x}\left(a_1(x,\tau)\frac{\partial\psi^N}{\partial x}\bar{\psi}^N + a_1(x,\tau)\frac{\partial\bar{\psi}^N}{\partial x}\psi^N\right)dxd\tau +$$
$$+2i\int_{\Omega_z} v_1(\tau)|\psi^N|^2 dxd\tau + 2i\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^4 dxd\tau = 2i\int_{\Omega_z}\mathrm{Im}(f\psi^N)dhd\tau, \forall z \in [0;L]$$

using the differentiability of the function $a_1(x,z)$, we can write the following equation from the last equation $\forall z \in [0,L]$

$$\int_{a_x}\frac{\partial}{\partial z}|\psi^N|^2 dxd\tau + \int_{a_x}\frac{\partial}{\partial x}(a_1(x,\tau)|\psi^N|^2)dxd\tau + 2\mathrm{Im}a_2\int_{\mathbf{a}_z}|\psi^N|^4 dxd\tau = \tag{3.12}$$
$$= \int_{\Omega_x}\frac{\partial a_1(x,\tau)}{\partial x}|\psi^N|^2 dxd\tau - 2\int_{\Omega_x} v_1(\tau)|\psi^N|^2 dxd\tau + 2\int_{\Omega_x}\mathrm{Im}(f\bar{\psi}^N)dxd\tau$$

it is clear that the second term on the left hand side of this equation is equal to zero since the function $\psi_N(x,z)$ satisfies the conditions $\psi^N(0,z) = \psi^N(l,z) = 0, z \in (0,L)$.

Therefore, using the conditions satisfied by the coefficients, we can easily obtain the following inequality from equation (3.12):

$$\|\psi^N(,,z)\|^2_{L_2(0,l)} + 2\mathrm{Im}a_2 \int_{\Omega_x} |\psi^N|^4 dxd\tau \leqslant \|\psi^N(,,0)\|^2_{L_2(0,l)} + \|f\|^2_{L_2(a)} + \tag{3.13}$$

$$+(\mu_5 + 2b_1 + 1) \int_0^z \|\psi^N(-\tau)\|^2_{L_2(0,l)} d\tau \forall t \in [0,T]$$

Using the formula (3.8) we can write the following relation:

$$\|\psi^N(.,0)\|^2_{L_2(0,l)} = \sum_{k=1}^{N} |c_k^N(0)|^2 \leqslant \sum_{k=1}^{\infty} |\varphi_k|^2 = |\varphi|^2_{L_2(0,\lambda)} \tag{3.14}$$

with the help of this relation, we obtain the following inequality from the inequality (3.13) :

$$\|\psi^N(.,z)\|^2_{L_2(0,l)} + 2\mathrm{Im}a_2 \int_{\omega_z} |\psi^N|^4 dxd\tau \leqslant \|\varphi\|^2_{L_2(0,\lambda)} + \|f\|^2_{L_2(a)} + \tag{3.15}$$

$$+(\mu_5 + 2b_1 + 1)\int_0^z \left\|\psi^N(.,\tau)\right\|^2_{L_2(0,l)} d\tau, \forall z \in [0,L].$$

since the second term on the left-hand side of this inequality is non-negative, we can write the following inequality:

$$\|\psi^N(.,z)\|^2_{L_2(0,l)} \leqslant \|\varphi\|^2_{L_2(0,\lambda)} + \|f\|^2_{L_2(a)} + (\mu_5 + 2b_1 + 1) \int_0^z \|\psi^N(-\tau)\|^2_{L_2(0,l)} d\tau \; \forall \in [0,L]$$

from this, with the help of Gronwall's lemma, we obtain that the following estimate holds:

$$\|\psi^N(,z)\|^2_{L_2(0,l)} \leqslant c_2\left(\|\varphi\|^2_{L_2(0,l)} + \|f\|^2_{L_2(\Omega)}\right) \forall z \in [0,L] \tag{3.16}$$

using this estimate, we also find the following estimate from inequality (3.15) $\forall z \in [0,L]$

$$\|\psi^N(,,z)\|^2_{L_2(0,l)} + 2\mathrm{Im}a_2 \int_{\Omega_z} |\psi^N|^4 dxd\tau \leqslant c_3\left(\|\varphi\|^2_{L_2(0,f)} + \|f\|^2_{L_2(\Omega)}\right) \tag{3.17}$$

Now let us try to evaluate the derivative of $\frac{\partial\psi^N}{\partial x}$. To this end, let us multiply the k^{th} equation of (3.9) by $\lambda_k c_k^{-N}(z)$, add the obtained equations from 1 to N over k and integrate them from zero to L over $z \leqslant L$.

We then obtain the following equality:

$$\int_{\Omega_z}\left(i\frac{\partial\psi^N}{\partial z}\Lambda\bar{\psi}^N - \left|\Lambda\psi^N\right|^2 + ia_1(x,\tau)\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N\right)dxd\tau +$$

$$+\int_{\Omega_z}\left(v_0(\tau)\psi^N\Lambda\bar{\psi}^N + iv_1(\tau)\psi^N\Lambda\bar{\psi}^N + a_2\left|\psi^N\right|^2\psi^N\Lambda\bar{\psi}^N\right)dxd\tau =$$

$$=\int_{\Omega_z} f(x,\tau)\Lambda\bar{\psi}^N(x,\tau)\,dxd\tau, \forall z\in[0,L]. \tag{3.18}$$

from this equation we deduce its complex conjugate and find the following equation:

$$\int_{\Omega_z}\left(i\left(\frac{\partial\psi^N}{\partial z}\Lambda\bar{\psi}^N + \frac{\partial\bar{\psi}^N}{\partial z}\Lambda\psi^N\right) + ia_1(x,\tau)\left(\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N + \frac{\partial\bar{\psi}^N}{\partial x}\Lambda\psi^N\right)\right)dxd\tau + \tag{3.19}$$

$$+\int_{\Omega_x}\left(v_0(\tau)(\psi^N\Lambda\bar{\psi}^N - \bar{\psi}^N\Lambda\psi^N) + iv_1(\tau)(\psi^N\Lambda\bar{\psi}^N + \bar{\psi}^N\Lambda\psi^N)\right)dxd\tau +$$

$$+\int_{a_x}(a_2|\psi^N|^2\psi^N\Lambda\bar{\psi}^N - \bar{a}_2|\psi^N|^2\bar{\psi}^N\Lambda\psi^N)dxd\tau =$$

$$=\int_{\mathbf{a}_x}\left(f(x,\tau)\Lambda\bar{\psi}^N(x,\tau) - \bar{f}(x,\tau)\Lambda\psi^N(x,\tau)\right)dxd\tau, \forall z\in[0,L]$$

using the formula (3.3) for the operator Λ and the partial integration formula, we can write the following equations with the help of the conditions $u_k(0) = u_k(l) = 0, k = 1,2\ldots$

$$\int_{\Omega_x} ia_1(x,\tau)\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N dxd\tau = \tag{3.20}$$

$$= -\int_{a_x} ia_0a_1(x,\tau)\frac{\partial\psi^N}{\partial x}\frac{\partial^2\bar{\psi}^N}{\partial x^2}dxd\tau + \int_{a_x} ia(x)a_1(x_s\tau)\frac{\partial\psi^N}{\partial x}\bar{\psi}^N dxd\tau$$

$$\int_{\mathbf{a}_x} (a_2|\psi^N|^2\psi^N\Pi\bar{\psi}^N - \bar{a}_2|\psi^N|^2\bar{\psi}^N\Lambda\psi^N)dxd\tau = \tag{3.21}$$

$$= 2i\mathrm{Im}a_2 \int_{\mathbf{\Omega}_x} a(x)|\psi^N|^4 dxad\tau +$$

$$+a_0\mathrm{Re}a_2 \int \left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} - \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau$$

Considering equations (3.20) - (3.24), we obtain the following equation from equation (3.19):

$$\int_{\Omega_x} (v_0(\tau)\psi^N\Lambda\bar{\psi}^N + iv_1(\tau)\psi^N\Lambda\bar{\psi}^N)dxd\tau = \tag{3.22}$$

$$= \int_{a_x} v_0(\tau)\psi^N\left(-a_0\frac{\partial^2\bar{\psi}^N}{\partial x^2} + a(x)\bar{\psi}^N\right) +$$

$$+iv_1(\tau)\psi^N\left(-a_0\frac{\partial^2\bar{\psi}^N}{\partial x^2} + a(x)\overline{\psi^N}\right)bhd\tau =$$

$$= \int_{a_x}\left(a_0v_0(\tau)\left|\frac{\partial\psi^N}{\partial x}\right|^2 + v_0(\tau)a(x)|\psi^N|^2\right)dxd\tau +$$

$$+\int_{a_x}\left(ia_0v_1(\tau)\left|\frac{\partial\psi^N}{\partial x}\right|^2 + iv_1(\tau)a(x)|\psi^N|^2\right)dxd\tau$$

$$\int_{\mathbf{a}_x} (a_2|\psi^N|^2\psi^N\Pi\bar{\psi}^N - \bar{a}_2|\psi^N|^2\bar{\psi}^N\Lambda\psi^N)dxd\tau = \tag{3.23}$$

$$= 2i\mathrm{Im}\,a_2 \int_{\Omega_x} a(x)|\psi^N|^4 dxad\tau +$$

$$+ia_0\mathrm{Im}\,a_2 \int \left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} + \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau +$$

$$+a_0\mathrm{Re}\,a_2 \int \left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} - \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau$$

$$\int_{a_x} f(x,\tau)\Lambda\bar{\psi}^N(x,\tau)dxd\tau = \tag{3.24}$$

$$= \int_{a_x} f(x,\tau)\left(-a_0\frac{\partial^2\bar{\psi}^N(x,\tau)}{\partial x^2} + a(x)\overline{\psi^N}(x,\tau)\right)dxd\tau =$$

$$= \int_{a_x}\left(a_0\frac{\partial f(x,\tau)}{\partial x}\frac{\partial\bar{\psi}^N(x,\tau)}{\partial x} + a(x)f(x,\tau)\bar{\psi}^N(x,\tau)\right)dxd\tau, \forall z \in [0,L].$$

Considering equations (3.20) - (3.24), we obtain the following equation from equation (3.19):

$$i\int_{\Omega_z} a_0\left(\frac{\partial^2\psi^N}{\partial z\partial x}\frac{\partial\bar{\psi}^N}{\partial x} + \frac{\partial^2\bar{\psi}^N}{\partial z\partial x}\frac{\partial\psi^N}{\partial x}\right)dxd\tau + i\int_{\Omega_z} a(x)\left(\frac{\partial\psi^N}{\partial z}\bar{\psi}^N + \frac{\partial\bar{\psi}^N}{\partial z}\psi^N\right)dxd\tau - \tag{3.25}$$

$$-i\int_{\Omega_z} a_0a_1(x,\tau)\left(\frac{\partial\psi^N}{\partial x}\frac{\partial^2\bar{\psi}^N}{\partial x^2} + \frac{\partial\bar{\psi}^N}{\partial x}\frac{\partial^2\psi^N}{\partial x^2}\right)dxd\tau +$$

$$+i\int_{\Omega_z} a(x)a_1(x,\tau)\left(\frac{\partial\psi^N}{\partial x}\bar{\psi}^N + \frac{\partial\bar{\psi}^N}{\partial x}\psi^N\right)dxd\tau +$$

$$+2i\int_{\Omega_z}\left(a_0v_1(\tau)\left|\frac{\partial\psi^N}{\partial x}\right|^2 + v_1(\tau)a(x)|\psi^N|^2\right)dxd\tau + 2i\mathrm{Im}a_2\int_{\Omega_z} a(x)|\psi^N|^4dxd\tau +$$

$$+ia_0\mathrm{Im}a_2\int_{\Omega_z}\left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} + \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau +$$

$$+a_0\mathrm{Re}a_2\int_{\Omega_z}\left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} - \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau =$$

$$= 2i\int_{\Omega_z} a_0\mathrm{Im}\left(\frac{\partial f(x,\tau)}{\partial x}\frac{\partial\bar{\psi}^N(x,\tau)}{\partial x}\right)dxd\tau +$$

$$+2i\int_{\Omega_z} a(x)\mathrm{Im}\left(f(x,\tau)\bar{\psi}^N(x,\tau)\right)dxd\tau, \forall z \in [0,L].$$

It is evident that the following equations hold:

$$ia_0\mathrm{Im}a_2\int_{\Omega_z}\left[\frac{\partial}{\partial x}(|\psi^N|^2\psi^N)\frac{\partial\bar{\psi}^N}{\partial x} + \frac{\partial}{\partial x}(|\psi^N|^2\bar{\psi}^N)\frac{\partial\psi^N}{\partial x}\right]dxd\tau = \tag{3.26}$$

$$= i4a_0\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2\left|\frac{\partial\psi^N}{\partial x}\right|^2dxd\tau + i2a_0\mathrm{Im}a_2\int_{\Omega_t}\mathrm{Re}\left[(\psi^N)^2\left(\frac{\partial\bar{\psi}^N}{\partial x}\right)^2\right]dxd\tau$$

$$a_0 \mathrm{Re} a_2 \int_{\Omega_z} \left[\frac{\partial}{\partial x}(|\psi^N|^2 \psi^N) \frac{\partial \bar{\psi}^N}{\partial x} - \frac{\partial}{\partial x}(|\psi^N|^2 \bar{\psi}^N) \frac{\partial \psi^N}{\partial x}\right] dx d\tau$$

with the help of these equations, we can easily obtain the following equation from equation (3.25):

$$\begin{gathered}
\int_{\Omega_z} a_0 \frac{\partial}{\partial z}\left|\frac{\partial \psi^N}{\partial x}\right|^2 dx d\tau + \int_{\Omega_z} a(x) \frac{\partial}{\partial z}|\psi^N|^2 dx d\tau - \\
- \int_{\Omega_z} a_0 \frac{\partial}{\partial x}\left(a_1(x,\tau)\left|\frac{\partial \psi^N}{\partial x}\right|^2\right) dx d\tau + \int_{\Omega_z} a_0 \frac{\partial a_1(x,\tau)}{\partial x}\left|\frac{\partial \psi^N}{\partial x}\right|^2 dx d\tau + \\
+ \int_{\Omega_z} \frac{\partial}{\partial x}(a(x) a_1(x,\tau)|\psi^N|^2) dx d\tau - \int_{\Omega_z} \frac{\partial}{\partial x}(a(x) a_1(x,\tau))|\psi^N|^2 dx d\tau + \\
+2 \int_{\Omega_z}\left(a_0 v_1(\tau)\left|\frac{\partial \psi^N}{\partial x}\right|^2 + v_1(\tau) a(x)|\psi^N|^2\right) dx d\tau + 2 \mathrm{Im} a_2 \int_{\Omega_z} a(x)|\psi^N|^4 dx d\tau \\
+4 a_0 \mathrm{Im} a_2 \int_{\Omega_z} |\psi^N|^2 \left|\frac{\partial \psi^N}{\partial x}\right|^2 dx d\tau + 2 a_0 \mathrm{Im} a_2 \int_{\Omega_z} \mathrm{Re}\left[(\psi^N)^2 \left(\frac{\partial \bar{\psi}^N}{\partial x}\right)^2\right] dx d\tau + \\
+2 a_0 \mathrm{Re} a_2 \int_{\Omega_z} \mathrm{Im}\left[(\bar{\psi}^N)^2 \left(\frac{\partial \psi^N}{\partial x}\right)^2\right] dx d\tau = \\
= 2 \int_{\Omega_z} a_0 \mathrm{Im}\left(\frac{\partial f(x,\tau)}{\partial x} \frac{\partial \bar{\psi}^N(x,\tau)}{\partial x}\right) dx d\tau + \\
+2 \int_{\Omega_z} a(x) \mathrm{Im}\left(f(x,\tau) \bar{\psi}^N(x,\tau)\right) dx d\tau, \forall z \in [0, L].
\end{gathered} \tag{3.28}$$

The third and fifth terms on the left side of this equation are equal to zero due to the conditions $a_1(0,z) = a_1(l,z) = 0, z \in (0,L)$. Taking this into consideration, we can write the equation (3.28) in the following form:

$$\int_{\Omega_z} a_0 \frac{\partial}{\partial z}\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau + \int_{\Omega_z} a(x)\frac{\partial}{\partial z}|\psi^N|^2 dxd\tau + 2\mathrm{Im}a_2\int_{\Omega_z} a(x)|\psi^N|^4 dxd\tau +$$

$$+4a_0\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau = -\int_{\Omega_z} a_0\frac{\partial a_1(x,\tau)}{\partial x}\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau +$$

$$+\int_{\Omega_z}\frac{\partial}{\partial x}(a(x)a_1(x,\tau))|\psi^N|^2 dxd\tau - 2\int_{\Omega_z}\left(a_0 v_1(\tau)\left|\frac{\partial\psi^N}{\partial x}\right|^2 + v_1(\tau)a(x)|\psi^N|^2\right)dxd\tau -$$

$$-2a_0\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}\left[(\psi^N)^2\left(\frac{\partial\bar{\psi}^N}{\partial x}\right)^2\right]dxd\tau - 2a_0\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}\left[(\bar{\psi}^N)^2\left(\frac{\partial\psi^N}{\partial x}\right)^2\right]dxd\tau +$$

$$+2\int_{\Omega_z} a_0\mathrm{Im}\left(\frac{\partial f(x,\tau)}{\partial x}\frac{\partial\bar{\psi}^N(x,\tau)}{\partial x}\right)dxd\tau + 2\int_{\Omega_z} a(x)\mathrm{Im}(f(x,\tau)\bar{\psi}^N(x,\tau))dxd\tau, \forall z\in[0,L]$$

from this equality we easily obtain the following inequality:

$$a_0\int_0^l\left|\frac{\partial\psi^N(x,z)}{\partial x}\right|^2 dx + \int_0^l a(x)|\psi^N(x,z)|^2 dx + 2\mathrm{Im}a_2\int_{\Omega_z} a(x)|\psi^N|^4 dxd\tau +$$

$$+4a_0\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau \leqslant a_0\int_0^l\left|\frac{\partial\psi^N(x,0)}{\partial x}\right|^2 dx + \int_0^l a(x)|\psi^N(x,0)|^2 dx +$$

$$+a_0\iint_{\Omega_z}\left|\frac{\partial a_1(x,\tau)}{\partial x}\right|\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau + \int_{\Omega_z}\left(a(x)\left|\frac{\partial a_1(x,\tau)}{\partial x}\right| + \left|\frac{da(x)}{dx}\right| a_1(x,\tau)\right)|\psi^N|^2 dxd\tau +$$

$$+2\int_{\Omega_z}\left(a_0|v_1(\tau)|\left|\frac{\partial\psi^N}{\partial x}\right|^2 + a(x)|v_1(\tau)||\psi^N|^2\right)dxd\tau +$$

$$+2a_0(\mathrm{Im}a_2 + |\mathrm{Re}a_2|)\int_{\Omega_z}|\psi^N|^2\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau + 2a_0\iint_{\Omega_z}\left|\frac{\partial f(x,\tau)}{\partial x}\right|\left|\frac{\partial\psi^N(x,\tau)}{\partial x}\right| dxd\tau +$$

$$+2\int_{\Omega_z} a(x)|f(x,\tau)||\psi^N(x,\tau)|dxd\tau, \forall z\in[0,L]$$

from this inequality, using the Cauchy-Bujakowski inequality with the help of the condition (2.4) and the conditions satisfied by the coefficients of the equation, we get that the following inequality holds:

$$a_0 \left\| \frac{\partial \psi^N(.,z)}{\partial x} \right\|^2_{L_2(0,l)} + a_0 \mathrm{Im} a_2 \int_{\Omega_z} |\psi^N|^2 \left| \frac{\partial \psi^N}{\partial x} \right|^2 dx d\tau \leqslant a_0 \left| \frac{\partial \psi^N(.,0)}{\partial x} \right\|^2_{L_2(0,l)} + \mu_1 \|\psi^N(.,0)\|^2_{L_2(0,l)} +$$

$$+a_0(\mu_5 + 2b_1 + 1) \int_{\Omega_z} \left| \frac{\partial \psi^N}{\partial x} \right|^2 dx d\tau + (\mu_2\mu_4 + \mu_1\mu_5 + 2\mu_1 b_1 + \mu_1) \int \left|\psi^N_{\Omega_z}\right|^2 dx d\tau +$$

$$+a_0 \int_{\Omega_z} \left| \frac{\partial f(x,\tau)}{\partial x} \right|^2 dx d\tau + \mu_1 \int |f(x,\tau)|^2_{\Omega_z} dx d\tau, \forall z \in [0,L]. \quad (3.29)$$

With the help of the formula (3.8) we easily obtain the following inequality:

$$\left\| \frac{\partial \psi^N(.,0)}{\partial x} \right\|^2_{L_2(0,l)} \leqslant c_4 \|\varphi\|^2_{W_2(0,l)} \quad (3.30)$$

with the help of this inequality, the relation (3.14) and the estimation (3.17), we can write the following inequality from inequality (3.29):

$$\left\| \frac{\partial \psi^N(.,z)}{\partial x} \right\|^2_{L_2(0,l)} + 2\frac{\mu_0}{a_0} \mathrm{Im} a_2 \int_{\Omega_z} |\psi^N|^4 dx d\tau + \mathrm{Im} a_2 \int_{\Omega_z} |\psi^N|^2 \left| \frac{\partial \psi^N}{\partial x} \right|^2 dx d\tau \leqslant \quad (3.31)$$

$$\leqslant c_5 \left(\|\varphi\|^2_{W^1_{W_2(0,l)}} + \|f\|^2_{W_2(\Omega)} \right) + c_6 \int_0^z \left\| \frac{\partial \psi^N(.,\tau)}{\partial x} \right\|^2_{L_2(0,l)} d\tau, \forall z \in [0,L].$$

Since the second and third terms on the left-hand side of this inequality are nonnegative, we obtain the following inequality:

$$\left\| \frac{\partial \psi^N(.,z)}{\partial x} \right\|^2_{L_2(0,l)} \leqslant c_7 \left(\|\varphi\|^2_{W^1_{W_2(0,l)}} + \|f\|^2_{W_2(\Omega)} \right) + c_8 \int_0^z \left\| \frac{\partial \psi^N(.,\tau)}{\partial x} \right\|^2_{L_2(0,l)} d\tau, \forall z \in [0,L].$$

Hence, with the help of Gronwall's Lemma, we derive that the following prediction holds:

$$\left\| \frac{\partial \psi^N(.,z)}{\partial x} \right\|^2_{L_2(0,l)} \leqslant c_9 \left(\|\varphi\|^2_{W_2(0,l)} + \|f\|^2_{W_{W_2}(\Omega)} \right), \forall z \in [0,L]. \quad (3.32)$$

with the help of this estimation, we can also write the following estimation from inequality (3.31):

$$\left\|\frac{\partial\psi^N(.,z)}{\partial x}\right\|^2_{L_2(0,l)} + 2\frac{\mu_0}{a_0}\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^4 dxd\tau + \mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau \leqslant \quad (3.33)$$
$$\leqslant c_{10}\left(\|\varphi\|^2_{W_2(0,l)} + \|f\|^2_{W_2(\Omega)}\right), \forall z \in [0,L].$$

Now let us investigate the derivative of $\frac{\partial^2\psi^N}{\partial x^2}$.For this purpose, let us multiply the k-th equation of (3.9) by $\lambda_k^2\bar{c}_k^N(z)$ and sum the obtained equations over k from $k=1$ to $k=N$. Then we get the following equation:

$$i\sum_{k=1}^N \int_0^l \frac{\partial\psi^N(x,z)}{\partial z}u_k(x)dx\lambda_k^2\bar{c}_k^N(z) - \sum_{k=1}^N \int_0^l \Lambda\psi^N(x,z)u_k(x)dx\lambda_k^2\bar{c}_k^N(z) +$$
$$+i\sum_{k=1}^N \int_0^l a_1(x,z)\frac{\partial\psi^N(x,z)}{\partial x}u_k(x)dx\lambda_k^2\bar{c}_k^N(z) + \sum_{k=1}^N \int_0^l v_0(z)\psi^N(x,z)u_k(x)dx\lambda_k^2\bar{c}_k^N(z) +$$
$$+i\sum_{k=1}^N \int_0^l v_1(z)\psi^N(x,z)u_k(x)dx\lambda_k^2\bar{c}_k^N(z) + \sum_{k=1}^N \int_0^l a_2|\psi^N(x,z)|^2\psi^N(x,z)u_k(x)dx\lambda_k^2\bar{c}_k^N(z) =$$
$$= \sum_{k=1}^N \int_0^l f(x,z)u_k(x)dx\lambda_k^2\bar{c}_k^N(z)$$

with the help of the relation $\Lambda u_k = \lambda_k u_k$ we can write the following equation using the partial integration formula:

$$i\sum_{k=1}^N \int_0^l \frac{\partial}{\partial z}(\Lambda\psi^N(x,z))u_k(x)dx\lambda_k\bar{c}_k^N(z) - \sum_{k=1}^N \int_0^l \Lambda(\Lambda\psi^N(x,z))u_k(x)dx\lambda_k\bar{c}_k^N(z) +$$
$$+i\sum_{k=1}^N \int_0^l \Lambda\left(a_1(x,z)\frac{\partial\psi^N(x,z)}{\partial x}\right)u_k(x)dx\lambda_k\bar{c}_k^N(z) + \sum_{k=1}^N \int_0^l v_0(z)\Lambda\psi^N(x,z)u_k(x)dx\lambda_k\bar{c}_k^N(z) +$$
$$+i\sum_{k=1}^N \int_0^l v_1(z)\Lambda\psi^N(x,z)u_k(x)dx\lambda_k\bar{c}_k^N(z) + \sum_{k=1}^N \int_0^l \Lambda(a_2|\psi^N(x,z)|^2\psi^N(x,z))u_k(x)dx\lambda_k\bar{c}_k^N(z) =$$
$$= \sum_{k=1}^N \int_0^l \Lambda f(x,z)u_k(x)dx\lambda_k\bar{c}_k^N(z)$$

Let us still use the relation $\Lambda u_k = \lambda_k u_k$ and the formula (3.8) to integrate the resulting equation over z from zero to $z \leqslant L$. Then we obtain the following equation:

$$i\int_{\Omega_z} \frac{\partial}{\partial z}(\Lambda\psi^N)\Lambda\bar{\psi}^N dxd\tau - \int_{\Omega_z} \Lambda(\Lambda\psi^N)\Lambda\bar{\psi}^N dxd\tau + i\int_{\Omega_z} \Lambda\left(a_1(x,\tau)\frac{\partial\psi^N}{\partial x}\right)\Lambda\bar{\psi}^N dxd\tau +$$
$$+\int_{\Omega_z} v_0(\tau)|\Lambda\psi^N|^2 dxd\tau + i\int_{\Omega_z} v_1(\tau)|\Lambda\psi^N|^2 dxd\tau + \int_{\Omega_z} \Lambda(a_2|\psi^N|^2\psi^N)\Lambda\bar{\psi}^N dxd\tau = \quad (3.34)$$
$$= \int_{\Omega_z} \Lambda f(x,\tau)\Lambda\bar{\psi}^N(x,\tau)dxd\tau, \forall z \in [0,L]$$

Let us try to transform the second, third and sixth terms on the left-hand side of this equation. In this case

$$\Lambda(\Lambda\psi^N) = -a_0\frac{\partial}{\partial x^2}(\Lambda\psi^N) + a(x)\Lambda\psi^N \quad (3.35)$$

using the formula, we can write the following equation with the help of the partial integration formula:

$$\int_{\Omega_z} \Lambda(\Lambda\psi^N)\Lambda\bar{\psi}^N dxd\tau = \int_{\Omega_z} a_0\left|\frac{\partial}{\partial x}(\Lambda\psi^N)\right|^2 dxd\tau + \int_{\Omega_z} a(x)|\Lambda\psi^N|^2 dxd\tau, \forall z \in [0,L]. \quad (3.36)$$

In order to transform the third term on the left-hand side of equation (3.34), first transform $\Lambda\left(a_1(x,z)\frac{\partial\psi^N}{\partial x}\right)$. With the help of the formula for the operator Λ we obtain the following equation:

$$\Lambda\left(a_1(x,z)\frac{\partial\psi^N}{\partial x}\right) = -a_0\frac{\partial^2}{\partial x^2}\left(a_1(x,z)\frac{\partial\psi^N}{\partial x}\right) + a(x)a_1(x,z)\frac{\partial\psi^N}{\partial x} =$$
$$= -a_0\frac{\partial^2 a_1(x,z)}{\partial x^2}\frac{\partial\psi^N}{\partial x} - 2a(x)\frac{\partial a_1(x,z)}{\partial x}\psi^N - a_1(x,z)\frac{da(x)}{dx}\psi^N +$$
(3.37)
$$+2\frac{\partial a_1(x,z)}{\partial x}\Lambda\psi^N + a_1(x,z)\frac{\partial}{\partial x}(\Lambda\psi^N).$$

Now let us transform the sixth term on the left side of equation (3.34). We can still write the following equation with the help of the formula for the operator Λ :

$$\Lambda(a_2|\psi^N|^2\psi^N) = -a_0\frac{\partial^2}{\partial x^2}(a_2|\psi^N|^2\psi^N) + a_2 a(x)|\psi^N|^2\psi^N = \quad (3.38)$$

$$= 2a_2\left|\psi^N\right|^2 \Lambda\psi^N - a_2 a(x)\left|\psi^N\right|^2\psi^N - 4a_0a_2\left|\frac{\partial\psi^N}{\partial x}\right|^2\psi^N - 2a_0a_2\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N$$

Thus, with the assistance of equations (3.36) - (3.38), we obtain the following equation from equation (3.34):

$$i\int_{\Omega_z}\frac{\partial}{\partial z}(\Lambda\psi^N)\Lambda\bar{\psi}^N dxd\tau - \int_{\Omega_z} a_0\left|\frac{\partial}{\partial x}(\Lambda\psi^N)\right|^2 dxd\tau - \int_{\Omega_z} a(x)|\Lambda\psi^N|^2 dxd\tau - \tag{3.39}$$
$$-i\int_{\Omega_z} a_0\frac{\partial^2 a_1(x,\tau)}{\partial x^2}\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N dxd\tau - i\int_{\Omega_z}\left(2a(x)\frac{\partial a_1(x,\tau)}{\partial x} + a_1(x,\tau)\frac{da(x)}{dx}\right)\psi^N\Lambda\bar{\psi}^N dxd\tau +$$
$$+i\int_{\Omega_z} 2\frac{\partial a_1(x,\tau)}{\partial x}|\Lambda\psi^N|^2 dxd\tau + i\int_{\Omega_z} a_1(x,\tau)\frac{\partial}{\partial x}(\Lambda\psi^N)\Lambda\bar{\psi}^N dxd\tau +$$
$$+\int_{\Omega_z} v_0(\tau)|\Lambda\psi^N|^2 dxd\tau + i\int_{\Omega_z} v_1(\tau)|\Lambda\psi^N|^2 dxd\tau + 2a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2 dxd\tau +$$
$$+2a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2 dxd\tau - a_2\int_{\Omega_z} a(x)|\psi^N|^2\psi^N\Lambda\bar{\psi}^N dxd\tau - 4a_0a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\psi^N\Lambda\bar{\psi}^N dxd\tau -$$
$$-2a_0a_2\int_{\Omega_z}\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N dxd\tau = \int_{\Omega_z}\Lambda f(x,\tau)L\bar{\psi}^N(x,\tau)dxd\tau, \forall z\in[0,L]$$

if we subtract its complex conjugate from this equation, we find the following equation:

$$i\iint_{\Omega_z}\left(\frac{\partial}{\partial z}(\Lambda\psi^N)\Lambda\bar{\psi}^N+\frac{\partial}{\partial z}(\Lambda\bar{\psi}^N)\Lambda\psi^N\right)dxd\tau+ \tag{3.40}$$
$$+4i\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|\Lambda\psi^N|^2dxd\tau+i\int_{\Omega_z}a_1(x,\tau)\left(\frac{\partial}{\partial x}(\Lambda\psi^N)\Lambda\bar{\psi}^N+\frac{\partial}{\partial x}(\Lambda\bar{\psi}^N)\Lambda\psi^N\right)dxd\tau-$$
$$-i\int_{\Omega_z}a_0\frac{\partial^2 a_1(x,\tau)}{\partial x^2}\left(\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N+\frac{\partial\bar{\psi}^N}{\partial x}\Lambda\psi^N\right)dxd\tau-$$
$$-i\int_{\Omega_z}\left(2a(x)\frac{\partial a_1(x,\tau)}{\partial x}+a_1(x,\tau)\frac{da(x)}{dx}\right)(\psi^N\Lambda\bar{\psi}^N+\bar{\psi}^N\Lambda\psi^N)dxd\tau+$$
$$+2i\int_{\Omega_z}v_1(\tau)|\Lambda\psi^N|^2dxd\tau+4i\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau-$$
$$-2i\mathrm{Im}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-2i\mathrm{Re}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-8ia_0\mathrm{Im}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-8ia_0\mathrm{Re}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-4ia_0\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau-4ia_0\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau=$$
$$=2i\int_{\Omega_z}\mathrm{Im}(\Lambda f(x,\tau)\Lambda\bar{\psi}^N(x,\tau))dxd\tau,\forall z\in[0,L].$$

Now let us transform the third term on the left-hand side of this equation. It is clear that the following equation holds:

$$i\int_{\Omega_z}a_1(x,\tau)\left(\frac{\partial}{\partial x}(\Lambda\psi^N)\Lambda\bar{\psi}^N+\frac{\partial}{\partial x}(\Lambda\bar{\psi}^N)\Lambda\psi^N\right)dxd\tau=i\int_{\Omega_z}\frac{\partial}{\partial x}(a_1(x,\tau)|\Lambda\psi^N|^2)dxd\tau- \tag{3.41}$$
$$-i\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|\Lambda\psi^N|^2dxd\tau$$

if we consider this equation on the left hand side of equation (3.40), we can write the following equation:

$$i\int_{\Omega_z}\frac{\partial}{\partial z}|\Lambda\psi^N|^2dxd\tau+i\int_{\Omega_z}\frac{\partial}{\partial x}(a_1(x,\tau)|\Lambda\psi^N|^2)dxd\tau+$$
$$+3i\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|\Lambda\psi^N|^2dxd\tau-2i\int_{\Omega_z}a_0\frac{\partial^2a_1(x,\tau)}{\partial x^2}\mathrm{Re}\left(\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N\right)dxd\tau-$$
$$-2i\int_{\Omega_z}\left(2a(x)\frac{\partial a_1(x,\tau)}{\partial x}+a_1(x,\tau)\frac{da(x)}{dx}\right)\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau+$$
$$+2i\int_{\Omega_z}v_1(\tau)|\Lambda\psi^N|^2dxd\tau+4i\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau-$$
$$-2i\mathrm{Im}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-2i\mathrm{Re}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-8ia_0\mathrm{Im}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-8ia_0\mathrm{Re}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-4ia_0\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau-4ia_0\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau=$$
$$=2i\int_{\Omega_z}\mathrm{Im}\left(\Lambda f(x,\tau)\Lambda\overline{\psi^N}(x,\tau)\right)dxd\tau,\forall z\in[0,L].$$

According to the conditions $a_1(0,z)=a_1(l,z)=0, z\in(0,L)$, the second term on the left side of this equation is equal to zero. Therefore, from the last equation we obtain the following equality:

$$\int_0^1|\Lambda\psi^N(x,z)|^2dx-\int_0^1|\Lambda\psi^N(x,0)|^2dx+$$
$$+3\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|\Lambda\psi^N|^2dxd\tau-2\int_{\Omega_z}a_0\frac{\partial^2a_1(x,\tau)}{\partial x^2}\mathrm{Re}\left(\frac{\partial\psi^N}{\partial x}\Lambda\bar{\psi}^N\right)dxd\tau-$$
$$-2\int_{\Omega_z}\left(2a(x)\frac{\partial a_1(x,\tau)}{\partial x}+a_1(x,\tau)\frac{da(x)}{dx}\right)\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau+$$
$$+2\int_{\Omega_z}v_1(\tau)|\Lambda\psi^N|^2dxd\tau+4\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau-$$
$$-2\mathrm{Im}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-2\mathrm{Re}a_2\int_{\Omega_z}a(x)|\psi^N|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-8a_0\mathrm{Im}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Re}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-8a_0\mathrm{Re}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2\mathrm{Im}(\psi^N\Lambda\bar{\psi}^N)dxd\tau-$$
$$-4a_0\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau-4a_0\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}\left(\left(\frac{\partial\psi^N}{\partial x}\right)^2\bar{\psi}^N\Lambda\bar{\psi}^N\right)dxd\tau=$$
$$=2\int_{\Omega_z}\mathrm{Im}(\Lambda f(x,\tau)\Lambda\bar{\psi}^N(x,\tau))dxd\tau,\forall z\in[0,L].$$

Using the formula $\Lambda\psi^N(x,0) = \sum_{k=1}^{N} c_k^N(0)\Lambda u_k(x)$ and the condition (3.10) we can write the following equation

$$\Lambda\psi^N(x,0) = \sum_{k=1}^{N} c_k^N(0)\Lambda u_k(x) = \sum_{k=1}^{N} \varphi_k \Lambda u_k(x) = \sum_{k=1}^{N} \lambda_k \varphi_k u_k(x) \tag{3.42}$$

on the other hand, it is also clear that the following equation holds:

$$\sum_{k=1}^{N} \lambda_k \varphi_k u_k(x) = \sum_{k=1}^{N} \lambda_k \int_0^l \varphi(\xi) u_k(\xi) d\xi u_k(x) = \sum_{k=1}^{N} \int_0^l \varphi(\xi)(x)\Lambda u_k(\xi) d\xi u_k(x)$$

with the help of the partial integration formula, we obtain the following equation:

$$\sum_{k=1}^{N} \lambda_k \varphi_k u_k(x) = \sum_{k=1}^{N} \int_0^1 \Lambda\varphi(\xi)(x) u_k(\xi) d\xi u_k(x)$$

taking this equation into account in equation (3.43), we obtain the following equation:

$$\Lambda\psi^N(x,0) = \sum_{k=1}^{N} \int_0^l \Lambda\varphi(\xi)(x) u_k(\xi) d\xi u_k(x) = \sum_{k=1}^{N} (\Lambda\varphi)_k u_k(x) \tag{3.43}$$

using this equality, we can write the following inequality:

$$\|\Lambda\psi^N(\cdot,0)\|_{L_2(0,l)}^2 = \sum_{k=1}^{N} |(\Lambda\varphi)_k|^2 \leqslant \sum_{k=1}^{\infty} |(\Lambda\varphi)_k|^2 = \|\Lambda\varphi\|_{L_2(0,l)}^2 \tag{3.44}$$

from this relation we can easily obtain the following inequality with the help of the operator Λ

$$\|\Lambda\psi^N(\cdot,0)\|_{L_2(0,l)}^2 \leqslant c_{11}\|\varphi\|_{W_2(0,l)}^2 \tag{3.45}$$

with the help of this inequality, we can conclude from equality (3.42) that the following inequality holds

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + 4\mathrm{Im}a_2 \int_{\Omega_z} |\psi^N|^2 |\Lambda\psi^N|^2 dxd\tau \leqslant c_{11}\|\varphi\|^2_{W_2(0,l)} + \tag{3.46}$$

$$+3\int_{\Omega_z} \left|\frac{\partial a_1(x,\tau)}{\partial x}\right| |\Lambda\psi^N|^2 dxd\tau + 2a_0 \int_{\Omega_z} \left|\frac{\partial^2 a_1(x,\tau)}{\partial x^2}\right| \left|\frac{\partial \psi^N}{\partial x}\right| |\Lambda\psi^N| dxd\tau +$$

$$+2\int_{\Omega_z} \left(2a(x)\left|\frac{\partial a_1(x,\tau)}{\partial x}\right| + |a_1(x,\tau)|\left|\frac{da(x)}{dx}\right|\right) |\psi^N||\Lambda\psi^N| dxd\tau + 2\int_{\Omega_t} |v_1(\tau)||L\psi^N|^2 dxd\tau +$$

$$+(2\mathrm{Im}a_2 + 2|\mathrm{Re}a_2|)\int_{\Omega_z} a(x)|\psi^N|^2|\psi^N||\Lambda\psi^N| dxd\tau +$$

$$+(12a_0\mathrm{Im}a_2 + 12a_0|\mathrm{Re}a_2|)\int_{\Omega_z} \left|\frac{\partial \psi^N}{\partial x}\right|^2 |\psi^N|\|\Lambda\psi^N| dxd\tau +$$

$$+2\int_{\Omega_z} |\Lambda f(x,\tau)||\Lambda\psi^N(x,\tau)| dxd\tau, \forall z \in [0,L].$$

Taking into account the conditions provided by the coefficients of the equation, we obtain the following inequality from the last inequality with the help of the CauchyBunjakowski inequality:

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + 4\mathrm{Im}a_2 \int_{\Omega_z} |\psi^N|^2 |\Lambda\psi^N|^2 dxd\tau \leqslant c_{11}\|\varphi\|^2_{W_2^2(0,l)} +$$

$$+(3\mu_5 + a_0\mu_6 + 2\mu_1\mu_5 + \mu_2\mu_4 + 2b_1 + 1)\int_{\Omega_z} |\Lambda\psi^N|^2 dxd\tau +$$

$$+a_0\mu_6 \iint_{\Omega_z} \left|\frac{\partial \psi^N}{\partial x}\right|^2 dxd\tau + (2\mu_1\mu_5 + \mu_2\mu_4)\int_{\Omega_z} |\psi^N|^2 dxd\tau +$$

$$+(\mathrm{Im}a_2 + |\mathrm{Re}a_2|)\mu_1^2 \int_{\Omega_z} |\psi^N|^4 dxd\tau + (\mathrm{Im}a_2 + |\mathrm{Re}a_2|)\int_{\Omega_z} |\psi^N|^2 |\Lambda\psi^N|^2 dxd\tau$$

$$+(12a_0\mathrm{Im}a_2 + 12a_0|\mathrm{Re}a_2|)\int_{\Omega_2} \left|\frac{\partial \psi^N}{\partial x}\right|^2 |\psi^N||\Lambda\psi^N| dxd\tau +$$

$$+\int_{\Omega_z} |Lf(x,\tau)|^2 dxd\tau, \forall z \in [0,L].$$

Hence, with the help of the condition (2.4) and the ε-Cauchy inequality, we can derive the following inequality:

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + \frac{5}{2}\mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau \leqslant c_{11}\|\varphi\|^2_{W_2^2(0,l)} +$$

$$+(3\mu_5 + a_0\mu_6 + 2\mu_1\mu_5 + \mu_2\mu_4 + 2b_1 + 1)\int_{\Omega_z}|\Lambda\psi^N|^2dxd\tau +$$

$$+a_0\mu_6\iint_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2dxd\tau + (2\mu_1\mu_5 + \mu_2\mu_4)\int_{\Omega_z}|\psi^N|^2dxd\tau +$$

$$+\frac{3}{2}\mathrm{Im}a_2\mu_1^2\int_{\Omega_z}|\psi^N|^4dxd\tau + 9a_0\mathrm{Im}a_2\varepsilon\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau +$$

$$+\frac{9a_0\mathrm{Im}a_2}{\varepsilon}\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^4dxd\tau + \int_{\Omega_z}|\Lambda f(x,\tau)|^2dxd\tau, \forall z \in [0,L].$$

By choosing from this inequality $\varepsilon = \frac{1}{6a_0}$ we find the following inequality:

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + \mathrm{Im}a_2\int_{\Omega_z}|\psi^N|^2|\Lambda\psi^N|^2dxd\tau \leqslant c_{11}\|\varphi\|^2_{W_2^2(0,l)} + \quad (3.47)$$

$$+(3\mu_5 + a_0\mu_6 + 2\mu_1\mu_5 + \mu_2\mu_4 + 2b_1 + 1)\int_{\Omega_z}|\Lambda\psi^N|^2dxd\tau +$$

$$+a_0\mu_6\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^2dxd\tau + (2\mu_1\mu_5 + \mu_2\mu_4)\int_{\Omega_z}|\psi^N|^2dxd\tau + \frac{3}{2}\mathrm{Im}a_2\mu_1^2\int_{\Omega_z}|\psi^N|^4dxd\tau +$$

$$+54a_0^2\mathrm{Im}a_2\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^4dxd\tau + \int_{\Omega_z}|\Lambda f(x,\tau)|^2dxd\tau, \forall z \in [0,L].$$

According to the inequality we know from the studies of (Ladyzhenskaya,1967 and Ladyzhenskaya,1973) we can write the following inequality:

$$\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^4dxd\tau \leqslant \beta\int_0^z\left\|\frac{\partial^2\psi^N(.,\tau)}{\partial x^2}\right\|_{L_2(0,l)}\left\|\frac{\partial\psi^N(.,\tau)}{\partial x}\right\|^3_{L_2(0,l)}d\tau, \forall z \in [0,L]. \quad (3.48)$$

Here $\beta > 0$ is a known constant. If we apply the Cauchy-Bunjakowski inequality to the right hand side of this inequality, we get the following inequality:

$$\int_{\Omega_z}\left|\frac{\partial\psi^N}{\partial x}\right|^4dxd\tau \leqslant \frac{\beta}{2}\int_0^z\left\|\frac{\partial^2\psi^N(.,\tau)}{\partial x^2}\right\|^2_{L_2(0,l)}d\tau + \frac{\beta}{2}\int_0^z\left\|\frac{\partial\psi^N(.,\tau)}{\partial x}\right\|^6_{L_2(0,l)}d\tau, \forall z \in [0,L] (3.49)$$

Taking this inequality into consideration, we get the following inequality from the inequality (3.47):

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + \mathrm{Im}a_2 \int_{\Omega_z} |\psi^N|^2 |\Lambda\psi^N|^2 dxd\tau \leqslant c_{11}\|\varphi\|^2_{W_2(0,l)} + \tag{3.50}$$

$$+(3\mu_5 + a_0\mu_6 + 2\mu_1\mu_5 + \mu_2\mu_4 + 2b_1 + 1)\int_{\Omega_z} |\Lambda\psi^N|^2 dxd\tau +$$

$$+a_0\mu_6 \iint_{\Omega_z} \left|\frac{\partial\psi^N}{\partial x}\right|^2 dxd\tau + (2\mu_1\mu_5 + \mu_2\mu_4)\int_{\Omega_z} |\psi^N|^2 dxd\tau + \frac{3}{2}\mathrm{Im}a_2\mu_1^2 \int_{\Omega_z} |\psi^N|^4 dxd\tau +$$

$$+27a_0^2\mathrm{Im}a_2\beta \int_0^z \left\|\frac{\partial\psi^N(.,\tau)}{\partial x}\right\|^6_{L_2(0,l)} d\tau + 27a_0^2\mathrm{Im}a_2\beta \int_0^2 \left\|\frac{\partial^2\psi^N(.,\tau)}{\partial x^2}\right\|^2_{L_2(0,l)} d\tau +$$

$$+\int_{\Omega} |\Lambda f(x,\tau)|^2 dxd\tau, \forall z \in [0,L].$$

It follows that the estimates of (3.17), (3.33) and

$$\|\Lambda f\|^2_{L_2(\Omega)} \leqslant c_{12}\|f\|^2_{W_{W_2}(\Omega)} \tag{3.51}$$

with the help of the inequality, we obtain the following inequality:

$$\|\Lambda\psi^N(\cdot,z)\|^2_{L_2(0,l)} + \mathrm{Im}a_2 \int_{\Omega_z} |\psi^N|^2 |\Lambda\psi^N|^2 dxd\tau \leqslant \tag{3.52}$$

$$+c_{13}\left(\|\varphi\|^2_{W_{W_2(0,l)}} + \|f\|^2_{W_2(\Omega)} + \|\varphi\|^6_{W_2(0,l)} + \|f\|^6_{\substack{W_2(\Omega)\\ W_{2,0}}}\right) + c_{14}\int_0^2 \int \frac{\partial^2\psi^N(.,\tau)}{\partial x^2} \|^2_{L_2(0,l)} d\tau +$$

$$+c_{15}\int_0^z \|\Lambda\psi^N(.,\tau)\|^2_{L_2(0,l)} d\tau, \forall z \in [0,L].$$

Using the formula for the operator Λ, we can write the following inequality:

$$\|\Lambda\psi^N(.,z)\|_{L_2(0,l)} = \left\|-a_0\frac{\partial^2\psi^N(.,z)}{\partial x^2} + a(.)\psi^N(.,z)\right\|_{L_2(0,l)} \geqslant$$

$$\geqslant a_0\left\|\frac{\partial^2\psi^N(.,z)}{\partial x^2}\right\|_{L_2(0,l)} - \|a(.)\psi^N(.,z)\|_{L_2(0,l)}$$

from this we deduce the following inequality

$$\left\|\frac{\partial^2\psi^N(.,z)}{\partial x^2}\right\|_{L_2(0,l)} \leqslant \frac{1}{a_0}\|\Lambda\psi^N(.,z)\|_{L_2(0,l)} + \frac{\mu_1}{a_0}\|\psi^N(.,z)\|_{L_2(0,l)} \tag{3.53}$$

considering this inequality and the estimation of (3.17), we obtain the following inequality from inequality (3.52):

$$\left\|\Lambda\psi^N(\cdot,z)\right\|^2_{L_2(0,l)}+\operatorname{Im}a_2\int\limits_{\Omega_z}\left|\psi^N\right|^2\left|\Lambda\psi^N\right|^2dxd\tau\le \tag{3.54}$$

$$\le c_{16}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)}+\|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)}+\|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right)+$$

$$+c_{17}\int\limits_0^z\left\|\Lambda\psi^N(.,\tau)\right\|^2_{L_2(0,l)}d\tau,\forall z\in[0,L].$$

Considering that the term on the left-hand side of this inequality is non-negative, we can write the following inequality:

$$\left\|\Lambda\psi^N(\cdot,z)\right\|^2_{L_2(0,l)}\le c_{16}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)}+\|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)}+\|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right)+$$

$$+c_{17}\int\limits_0^z\left\|\Lambda\psi^N(.,\tau)\right\|^2_{L_2(0,l)}d\tau,\forall z\in[0,L].$$

From this, with the assistance of Gronwall's lemma, we obtain that the following prediction holds:

$$\left\|\Lambda\psi^N(\cdot,z)\right\|^2_{L_2(0,l)}\le c_{18}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)}+\|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)}+\|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right),\ \forall z\in[0,L]. \tag{3.55}$$

Given this estimate, we also find that the following estimate from inequality (3.54) holds:

$$\left\|\Lambda\psi^N(\cdot,z)\right\|^2_{L_2(0,l)}\le c_{16}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)}+\|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)}+\|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right)+ \tag{3.56}$$

$$+c_{17}\int\limits_0^z\left\|\Lambda\psi^N(.,\tau)\right\|^2_{L_2(0,l)}d\tau,\forall z\in[0,L].$$

From the inequality (3.53) with the help of the estimates (3.17) and (3.55) we obtain the following estimate:

$$\left\|\Lambda\psi^N(\cdot,z)\right\|^2_{L_2(0,l)}+\operatorname{Im}a_2\int\limits_{\Omega_z}\left|\psi^N\right|^2\left|\Lambda\psi^N\right|^2dxd\tau\le \tag{3.57}$$

$$\le c_{19}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^{2}(0,l)}+\|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^{1}(0,l)}+\|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right),\forall z\in[0,L].$$

By adding the estimates (3.16), (3.32) and (3.57) side by side, we can write the following estimate:

$$\left\|\psi^N(.,z)\right\|^2_{\overset{0}{W}{}_2^2(0,l)} \leq c_{21}\left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(0,l)} + \|f\|^2_{\overset{0}{W}{}_2^{2,0}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^1(0,l)} + \|f\|^6_{\overset{0}{W}{}_2^{1,0}(\Omega)}\right), \forall z \in [0,L]. \tag{3.58}$$

Now let us evaluate the derivative of $\frac{\partial\psi^N}{\partial z}$. For this purpose, let us multiply the k-th equation of (3.9) by $\frac{d\bar{c}_k^N(z)}{dz}$ and sum the obtained equations over k from $k = 1$ to $k = N$. If we integrate both sides of the obtained equation over the interval $(0, L)$, we obtain the following equation:

$$\int_\Omega \left(i\left|\frac{\partial\psi^N}{\partial z}\right|^2 + a_0\frac{\partial^2\psi^N}{\partial x^2}\frac{\partial\bar{\psi}^N}{\partial z} + ia_1(x,z)\frac{\partial\psi^N}{\partial x}\frac{\partial\bar{\psi}^N}{\partial z} - a(x)\psi^N\frac{\partial\bar{\psi}}{\partial z}\right)dxdz +$$
$$+\int_\Omega \left(v_0(z)\psi^N\frac{\partial\bar{\psi}^N}{\partial z} + iv_1(z)\psi^N\frac{\partial\bar{\psi}^N}{\partial z} + a_2|\psi^N|^2\psi^N\frac{\partial\bar{\psi}^N}{\partial z}\right)dxdz = \int_\Omega f\frac{\partial\bar{\psi}^N}{\partial z}dxdz$$

from this we can derive the following equation:

$$\int_\Omega \left|\frac{\partial\psi^N}{\partial z}\right|^2 dxdz = -i\int_\Omega\left(-a_0\frac{\partial^2\psi^N}{\partial x^2} - ia_1(x,z)\frac{\partial\psi^N}{\partial x} + a(x)\psi^N\right)\frac{\partial\bar{\psi}^N}{\partial z}dxdz -$$
$$= -i\int_\Omega(-v_0(z)\psi^N - iv_1(z)\psi^N - a_2|\psi^N|^2\psi^N + f(x,z))\frac{\partial\bar{\psi}^N}{\partial z}dxdz$$

from this equation, with the help of the Cauchy-Bunjakowski inequality, we obtain the following inequality:

$$\int_\Omega \left|\frac{\partial\psi^N}{\partial z}\right|^2 dxdz \leqslant 7a_0^2\int_\Omega\left|\frac{\partial^2\psi^N}{\partial x^2}\right|^2 dxdz + 7\mu_4^2\int_\Omega\left|\frac{\partial\psi^N}{\partial x}\right|^2 dxdz + \tag{3.59}$$
$$+7(\mu_1^2 + b_0^2 + b_1^2)\int_\Omega|\psi^N|^2dxdz + 7|a_2|^2\int_\Omega|\psi^N|^6dxdz + 7\int_\Omega|f|^2dxdz$$

now let us consider the fourth term on the right hand side of this inequality. According to the inequality we know from (Lions,1972; Yagubov,Ibrahimov and Suleymanov,2022) we can write the following inequality:

$$\left\|\psi^N(.,z)\right\|_{L_\infty(0,l)} \le \tilde{\beta}\left\|\frac{\partial\psi^N(.,z)}{\partial x}\right\|_{L_2(0,l)}^{\frac{1}{2}}\left\|\psi^N(.,z)\right\|_{L_2(0,l)}^{\frac{1}{2}}, \forall z\in[0,L]\ . \tag{3.60}$$

Here $\tilde{\beta} > 0$ is a known constant. From this inequality, with the help of (3.16) and (3.32) we obtain the following estimate:

$$\left\|\psi^N(.,z)\right\|_{L_\infty(0,l)}^6 \le c_{22}\left(\|\varphi\|_{\overset{0}{W}{}_2^1(0,l)}^6 + \|f\|_{\overset{0}{W}{}_2^{1,0}(\Omega)}^6\right), \forall z\in[0,L] \tag{3.61}$$

With the help of this estimate and the estimate of (3.58), we find the following estimate from the inequality (3.59):

$$\left\|\frac{\partial\psi^N}{\partial z}\right\|_{L_2(\Omega)}^2 \le c_{23}\left(\|\varphi\|_{\overset{0}{W}{}_2^2(0,l)}^2 + \|f\|_{\overset{0}{W}{}_2^{2,0}(\Omega)}^2 + \|\varphi\|_{\overset{0}{W}{}_2^1(0,l)}^6 + \|f\|_{\overset{0}{W}{}_2^{1,0}(\Omega)}^6\right), \forall z\in[0,L]. \tag{3.62}$$

If we integrate both sides of the estimate (3.58) over the interval $(0,L)$ and add the resulting estimate with the estimate (3.62), we obtain the following estimate:

$$\left\|\psi^N\right\|_{W_2^{2,1}(\Omega)}^2 \le c_{24}\left(\|\varphi\|_{\overset{0}{W}{}_2^2(0,l)}^2 + \|f\|_{\overset{0}{W}{}_2^{2,0}(\Omega)}^2 + \|\varphi\|_{\overset{0}{W}{}_2^1(0,l)}^6 + \|f\|_{\overset{0}{W}{}_2^{1,0}(\Omega)}^6\right), \forall z\in[0,L],\ N=1,2,\ldots. \tag{3.63}$$

Here the constant $c_{24} > 0$ is independent of N. Using this estimate and choosing $c_0 = c_{24}$, we prove that the lemma holds. Lemma 3.2 is proved. Now let us continue the proof of the theorem. By (3.11) or (3.63) we can choose a subsequence $\{\psi^N(x,z)\}$ from the sequence $\{\psi^{N_m}(x,z)\}$ which converges to the function $\psi(x,z)$ in the space $\mathrm{W}_0{}^{2,1}$ $W_2(\Omega)$. Let us show that this limit function (2.1) - (2.3) is the solution of the initial boundary value problem in the sense of Definition 2.1. To this end, let us first show that the function $\psi(x,z)$ almost satisfies equation (2.1) for $(x,z)\in\Omega$.Therefore, when $N = N_m$(3.9), the k-th Fourier coefficient of any function $\bar{\eta}(x,z)$ belonging to the space $L_2(\Omega)$ is the k-th Fourier coefficient. Multiply by the function $\bar{\eta}_k(z) = (\bar{\eta}(.,z),u_k)_{L_2(0,l)}$. Let us multiply the obtained equations over k from $k=1$ to $N' \leqslant N_m$ and integrate over the interval $(0,L)$. Then for any function $\bar{\eta}_k^{N'}(x,z) = \sum_{k=1}^{N'} \bar{\eta}_k(z)u_k(x)N' \leqslant N_m$ we obtain the following integral equivalence:

$$\int_{\Omega}\left(i\frac{\partial\psi^{N_m}}{\partial z}+a_0\frac{\partial^2\psi^{N_m}}{\partial x^2}+ia_1(x,z)\frac{\partial\psi^{N_m}}{\partial x}-a(x)\psi^{N_m}+\right. \tag{3.64}$$
$$\left.+v_0(z)\psi^{N_m}+iv_1(z)\psi^{N_m}+a_2|\psi^{N_m}|^2\psi^{N_m}-f(x,z)\right)\bar{\eta}^{N'}(x,z)dxdz=0$$

from the compact embedding of the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ into the space $L_2(\Omega)$ we can write the following limit relation:

$$\text{For } m\to\infty, \psi^{N_m}\to\psi \text{ is strongly convergent in } L_2(\Omega) \tag{3.65}$$

if this is the case, we can choose a subsequence $\{\psi^{N_m}(x,z)\}$ which converges almost immediately to $\psi(x,z)$ in the Ω region. For simplicity, let us denote the convergent subsequence again by $\{\psi^{N_m}(x,z)\}$. Then we can express the following limit relation:

For $m\to\infty, |\psi^{N_m}(x,z)|^2\psi^{N_m}(x,z)\to|\psi(x,z)|^2\psi(x,z)$ (3.66), weakly in $L_2(\Omega)'$ for almost every $(x,z)\in\Omega$. On the other hand, by embedding the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ into the space $L_\infty(0,L{:}\,W_2^1(0,l))$ (see [25]), we obtain the following inequality:

$$\left\|\psi^{N_m}\right\|_{L_\infty\left(0,L;\overset{0}{W}{}_2^1(0,l)\right)}\le c_{25}\left\|\psi^{N_m}\right\|_{\overset{0}{W}{}_2^{2,1}(\Omega)}$$

then with the help of the estimation (3.11) we can write the following inequality when $N=N_m, m=1,2,\ldots$.

$$\left\|\psi^{N_m}\right\|_{L_\infty\left(0,L;\overset{0}{W}{}_2^1(0,l)\right)}\le c_{26}, m=1,2,\ldots \tag{3.67}$$

it is obvious that for $m=1,2,\ldots$ the following inequality applies:

$$\left\|\left|\psi^{N_m}\right|^2\psi^{N_m}\right\|_{L_2(\Omega)}^2=\int_0^L\left\|\left|\psi^{N_m}(.,z)\right|^2\psi^{N_m}(.,z)\right\|_{L_2(0,l)}^2dz\le\int_0^L\left\|\psi^{N_m}(.,z)\right\|_{L_6(0,l)}^6dz, m=1,2,\ldots \tag{3.68}$$

on the other hand, from the inequality (3.60) we obtain the following inequality when. $N=N_m, m=1,2,\ldots$:

$$\int_0^L \left\|\psi^{N_m}(.,z)\right\|_{L_6(0,l)}^6 dz \le l\int_0^L \left\|\psi^{N_m}(.,z)\right\|_{L_\infty(0,l)}^6 dz \le c_{26}\int_0^L \left\|\psi^{N_m}(.,z)\right\|_{\overset{0}{W}{}_2^1(0,l)}^6 dz \le$$

$$\le c_{27}\left\|\psi^{N_m}\right\|_{L_\infty\left(0,L;\overset{0}{W}{}_2^1(0,l)\right)}^6, m=1,2,\dots .$$

Then, with the help of this inequality and the inequality (3.67), we find from the inequality (3.68) that the following inequality holds:

$$\||\psi^{N_m}|^2\psi^{N_m}\|_{L_2(\Omega)}^2 \leqslant c_{28}, m = 1,2, \dots \tag{3.69}$$

therefore, based on this inequality and the lemma known from (Ladyzhenskaya,1967) we can conclude that the following limit relation holds:

For $m \to \infty$, $|\psi^{N_m}|^2\psi^{N_m} \to |\psi|^2\psi$, converges weakly in $L_2(\Omega)$. (3.70)

Taking into account this limit relation and the weak convergence of the subsequence $\psi^{N_m}(x,z)$ to the function $\psi(x,z)$ in the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ by passing to the limit in the integral identity (3.64) over $Nm, m = 1,2, \dots$, we obtain that the following integral identity holds for any function

$$\bar{\eta}k^{N'}(x,z) = \sum\ k = 1^{N'}\bar{\eta}k(z)uk(x): \int_\Omega \left(\psi(x,z)\frac{\partial\bar{\eta}k^{N'}}{\partial x} - \bar{\eta}k^{N'}(x,z)\frac{\partial\psi}{\partial x}\right)dxdz = 0$$

$$\int_\Omega \left(i\frac{\partial\psi}{\partial z} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1(x,z)\frac{\partial\psi}{\partial x} - a(x)\psi + \right.$$

$$\left. +v_0(z)\psi + iv_1(z)\psi + a_2|\psi|^2\psi - f(x,z)\right)\bar{\eta}^{N'}(x,z)dxdz = 0$$

$$\bar{\eta}_k^{N'}(x,z) = \sum_{k=1}^{N'} \bar{\eta}_k(z)u_k(x) \text{ is a partial sum of the series } \bar{\eta}(x,z) = \sum_{k=1}^{\infty} \bar{\eta}_k(z)u_k(x)$$

since N' For $\to \infty$ the partial sum $\bar{\eta}_k^{N'}(x,z)$ converges to the function $\bar{\eta}(x,z)$ in the space $L_2(\Omega)$. If This then in the last identity N' If we take the limit for $\to \infty$, we can conclude that the limit function $\psi(x,z)$ satisfies the following integral identity for any function $\eta = \eta(x,z)$ belonging to the space $L_2(\Omega)$:

$$\int_\Omega \left(i\frac{\partial\psi}{\partial z} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1(x,z)\frac{\partial\psi}{\partial x} - a(x)\psi + \right. \tag{3.71}$$

$$\left. +v_0(z)\psi + iv_1(z)\psi + a_2|\psi|^2\psi - f(x,z)\right)\bar{\eta}(x,z)dxdz = 0$$

From this, we obtain that the function $\psi(x,z)$ satisfies the equation (2.1) for almost every $(x,z) \in \Omega$.

Now let us show that the limit function $\psi(x,z)$ satisfies the initial value condition (2.2) for almost every $x \in (0,l)$, that is, the condition $\psi(x,0) = \varphi(x), \forall x \in (0,l)$. Taking into account that the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ is compactly embedded into the space $C^0([0,L], L_2(0,l))$, we can write the following limit relation, uniformly with respect to $z \in [0,L]$:

$$\text{For } m \to \infty, \|\psi^{N_m}(.,z) - \psi(.,z)\|_{L_2(0,l)} \to 0 \tag{3.72}$$

on the other hand, it is evident that the following inequality holds:

$$\|\psi(.,0) - \varphi\|_{L_2(0,l)} \leqslant \|\psi(.,0) - \psi^{N_m}(.,0)\|_{L_2(0,l)} + \|\psi^{N_m}(.,0) - \varphi\|_{L_2(0,l)} \tag{3.73}$$

when $z = 0$, using the limit relation (3.72), the first term on the right-hand side of this inequality approaches zero for $m \to \infty$. Therefore, let us show that the second term on the right-hand side of inequality (3.73) also approaches zero for $m \to \infty$. Using the formula (3.8) we can write the following formula:

$$\psi^{N_m}(x,0) = \sum_{k=1}^{N_m} c_k^{N_m}(0) u_k(x) = \sum_{k=1}^{N_m} \varphi_k u_k(x) = \varphi^{N_m}(x)$$

the $\varphi^{N_m}(x)$ function is the partial sum of the Fourier series of the function $\varphi = \varphi(x)$ belonging to the space $\overset{0}{W}{}_2^{2}(0,l)$. Taking this into account, if we take limit for $m \to \infty$ in the second term on the right-hand side of the inequality (3.73), we obtain the following limit relation

$$\text{For } m \to \infty, \|\psi^{N_m}(.,0) - \varphi\|_{L_2(0,l)} \to 0 \tag{3.74}$$

thus, if we take this limit relation and the limit relation (3.72) when $z = 0$ and limit for $m \to \infty$ on both sides of the inequality (3.73), we find the following relation:

$$\|\psi(.,0) - \varphi\|_{L_2(0,l)} = 0$$

from this, we obtain that the limit function $\psi(x,z)$ satisfies the initial value condition (2.2) for almost every $x \in (0,l)$, that is, the condition $\psi(x,0) = \varphi(x), \forall x \in (0,l)$. Finally, let us show that the limit function $\psi(x,z)$ satisfies the boundary value conditions (2.3) for almost every $z \in (0,L)$. From the compact embedding of the space $W_2^{2,1}(\Omega)$ into the space $C^0([0,l],L_2(0,L))$, we can write the following limit relation, uniformly with respect to $x \in [0,l]$:

$$\text{For } m \to \infty, \|\psi^{N_m}(x,.) - \psi(x,.)\|_{L_2(0,L)} \to 0, \forall x \in [0,l]. \tag{3.75}$$

On the other hand, it is apparent that the following inequality holds:

$$\|\psi(s,.)\|_{L_2(0,L)} \leqslant \|\psi(s,.) - \psi^{N_m}(s,)\big)\|_{L_2(0,L)} + \|\psi^{N_m}(s,)\|_{L_2(0,L)}, s = 0, l \tag{3.76}$$

if we use the limit relation (3.75) when $x = 0, x = l$, we can see that the first term on the right hand side of this inequality approaches zero for $m \to \infty$. Therefore, let us show that the second term on the right-hand side of the inequality (3.76) also approaches zero for $m \to \infty$. Using the formula (3.8), we can indeed write the following inequalities:

$$\psi^{N_m}(s,z) = \sum_{k=1}^{N_m} c_k^{N_m}(z) u_k(s), s = 0, l \tag{3.77}$$

if we consider the boundary value conditions $u_k(0) = u_k(l) = 0$, we obtain the following relations:

$$\psi^{N_m}(0,z) = \psi^{N_m}(l,z) = 0 \tag{3.78}$$

thus, if we use these relations and the limit relation (3.75) when $x = 0, x = l$ and limit for $m \to \infty$ on both sides of the inequality (3.76), we get the following relations:

$$\|\psi(s,.)\|_{L_2(0,L)} = 0, s = 0, l \tag{3.79}$$

from these relations, we obtain that the following boundary value conditions are valid:

$$\psi(0,z) = \psi(l,z) = 0, \overset{0}{\forall} z \in (0,L).$$

Thus, we have proved that the limit function $\psi(x,z)$ is a solution of the initial boundary value problem $(2.1)-(2.3)$ in the sense of Definition 2.1 belonging to the space $W_2(\Omega)$, and for this solution, the estimate (3.1) holds. Indeed, taking into account the convergence property of the subsequence $\psi^{N_m}(x,z)$ to the function $\psi(x,z)$ in the estimate (3.11), and setting $N=N_m, m=1,2,...$, we find that the estimate (3.1) holds by taking the limit as $m\to\infty$. Now, let us show that the solution of the initial boundary value problem (2.1) - (2.3) is unique. Suppose that $\psi(x,z)$ and $\phi(x,z)$ are any two solutions of the initial boundary value problem (2.1) - (2.3). Let $w(x,z)=\psi(x,z)-\phi(x,z)$. Then, it is clear that the function $w=w(x,z)$ is a solution of the following initial boundary value problem:

$$i\frac{\partial w}{\partial z}+a_0\frac{\partial^2 w}{\partial x^2}+ia_1(x,z)\frac{\partial w}{\partial x}-a(x)w+v_0(z)w+iv_1(z)w+ \quad (3.80)$$
$$+a_2(|\psi|^2+|\phi|^2)w+a_2\psi\phi\bar{w}=0,(x,z)\in\Omega$$
$$w(x,0)=0,x\in(0,l) \quad (3.81)$$
$$w(0,z)=w(l,z)=0,z\in(0,L) \quad (3.82)$$

Since the functions $\psi(x,z)$ and $\phi(x,z)$ are solutions of the initial boundary value problem $(2.1)-(2.3)$ belonging to the space $W^{02,1}(\Omega)$, the function $w=w(x,z)$ must satisfy the identity

$$\int_{\Omega_z}\left(i\frac{\partial w}{\partial z}+a_0\frac{\partial^2 w}{\partial x^2}+ia_1(x,\tau)\frac{\partial w}{\partial x}-a(x)w+v_0(\tau)w+iv_1(\tau)w+\right. \quad (3.83)$$
$$\left.+a_2(|\psi|^2+|\phi|^2)w+a_2\psi\phi\bar{w}\right)\bar{\eta}(x,\tau)dxd\tau=0,\forall z\in[0,L].$$

It is evident that $w(x,z)$ satisfies the integral identity and the initial condition (3.81) for almost every $x\in(0,l)$, and the boundary condition (3.82) for almost every $z\in(0,L)$. In this integral identity, let us take the function $w(x,z)$ belonging to the space $\overset{0}{W}{}_2^{2,1}(\Omega)$ instead of $\eta(x,z)$. Then, with the help of the partial integration formula, we obtain the following equality:

$$\int_{\Omega_z}\left(i\frac{\partial w}{\partial z}+a_0\frac{\partial^2 w}{\partial x^2}+ia_1(x,\tau)\frac{\partial w}{\partial x}-a(x)w+v_0(\tau)w+iv_1(\tau)w+\right.$$

(3.84)

$$\left.+a_2\left(|\psi|^2+|\phi|^2\right)w+a_2\psi\phi\bar{w}\right)\bar{\eta}(x,\tau)dxd\tau=0,\forall z\in[0,L]$$

if we subtract its complex conjugate from this equation, we can write the following equation:

$$\int_{\Omega_z}\left(\frac{\partial w}{\partial z}\bar{w}+\frac{\partial \bar{w}}{\partial z}w\right)dxd\tau+\int_{\Omega_z}\left(a_1(x,\tau)\frac{\partial w}{\partial x}\bar{w}+a_1(x,\tau)\frac{\partial \bar{w}}{\partial x}w\right)dxd\tau+2\int_{\Omega_z}v_1(\tau)|w|^2dxd\tau+$$
$$+2\mathrm{Im}a_2\int_{\Omega_z}(|\psi|^2+|\phi|^2)|w|^2dxd\tau+2\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}(\psi\phi(\bar{w})^2)dxd\tau+$$
$$+2\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}(\psi\phi(\bar{w})^2)dxd\tau=0,\forall z\in[0,L]. \tag{3.84}$$

If we transform the second term on the left hand side of this equation, we get the following equation:

$$\int_{\Omega_z}\left(a_1(x,\tau)\frac{\partial w}{\partial x}\bar{w}+a_1(x,\tau)\frac{\partial \bar{w}}{\partial x}w\right)dxd\tau$$
$$=\int_{\Omega_z}\frac{\partial}{\partial x}(a_1(x,\tau)|w|^2)dxd\tau-\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|w|^2dxd\tau$$

under the boundary value conditions of (3.82), the first term on the right hand side of this equation is equal to zero. We can therefore write the following equation:

$$\int_{\Omega_z}\left(a_1(x,\tau)\frac{\partial w}{\partial x}\bar{w}+a_1(x,\tau)\frac{\partial \bar{w}}{\partial x}w\right)dxd\tau=-\int_{\Omega_z}\frac{\partial a_1(x,\tau)}{\partial x}|w|^2dxd\tau$$

considering this equation, we obtain the following equation from equation (3.84):

$$\int_{\Omega_z}\left(\frac{\partial w}{\partial z}\bar{w}+\frac{\partial \bar{w}}{\partial z}w\right)dxd\tau+2\mathrm{Im}a_2\int_{\Omega_z}(|\psi|^2+|\phi|^2)|w|^2dxd\tau=$$
$$=-2\int_{\Omega_z}v_1(\tau)|w|^2dxd\tau-2\mathrm{Im}a_2\int_{\Omega_z}\mathrm{Re}(\psi\phi(\bar{w})^2)dxd\tau-2\mathrm{Re}a_2\int_{\Omega_z}\mathrm{Im}\big(\psi\phi(\bar{w})\big)dxd\tau,\forall z\in[0,L].$$

From this equation, and with the help of the initial value condition (3.81), we find that the following inequality holds:

$$\|w(.,z)\|^2_{L_2(0,l)}+2\mathrm{Im}a_2\int_{\Omega_z}(|\psi|^2+|\phi|^2)|w|^2dxd\tau\leqslant$$
$$\leqslant 2\int_{\Omega_z}|v_1(\tau)||w|^2dxd\tau+2(\mathrm{Im}a_2+|\mathrm{Re}a_2|)\int_{\Omega_z}|\psi||\phi||w|^2dxd\tau,\forall z\in[0,L].$$

Hence, from the condition for $2|\psi||\phi|\leqslant|\psi|^2+|\phi|^2$ and the function $v_1(z)$ we obtain the following inequality:

$$\|w(.,z)\|^2_{L_2(0,l)} + 2\mathrm{Im}a_2 \int_{\Omega_z} (|\psi|^2 + |\phi|^2)|w|^2 dxd\tau \leqslant$$
$$\leqslant 2b_1 \int_{\Omega_z} \left| w|^2 dxd\tau + (\mathrm{Im}a_2 + |\mathrm{Re}a_2|) \int_{\Omega_z} (|\psi|^2 + |\phi|^2) \right| w|^2 dxd\tau, \forall z \in [0, L].$$

From this inequality, with the condition (2.4), we find the following inequality:

$$\|w(.,z)\|^2_{L_2(0,l)} + \frac{1}{2}\mathrm{Im}a_2 \int_{\Omega_z} (|\psi|^2 + |\phi|^2)|w|^2 dxd\tau \leqslant 2b_1 \int_{\Omega_z} |w|^2 dxd\tau, \forall t \in [0, T].$$

Provided that the second term on the left-hand side of this equation is non-negative, we can write the following inequality:

$$\|w(.,z)\|^2_{L_2(0,l)} \leqslant 2b_1 \int_0^z \|w(.,\tau)\|^2_{L_2(0,l)} d\tau, \forall z \in [0, L].$$

Applying Gronwall's lemma, we obtain the following relation:

$$\|w(.,z)\|^2_{L_2(0,l)} = 0, \forall z \in [0, L].$$

From this we get the following relation:

$$w(x,z) = 0, \overset{0}{\forall} x \in (0,l), \forall z \in [0, L].$$

From this relation and the formula $w(x,z) = \psi(x,z) - \phi(x,z)$ it follows that the solution of the initial boundary value problem (2.1) - (2.3) is unique. Theorem 3.1 is proved.

Setting and Solution of the Second Kind of Initial Boundary Value Problem

Consider the following second kind of initial boundary value problem for finding the function $\psi = \psi(x,z)$:

$$i\frac{\partial\psi}{\partial z} + a_0\frac{\partial^2\psi}{\partial x^2} + ia_1(x,z)\frac{\partial\psi}{\partial x} - a(x)\psi + v_0(z)\psi + iv_1(z)\psi + a_2|\psi|^2 = f(x,z), (x,z) \in \Omega$$

$$\psi(x,0) = \varphi(x), x \in (0,l) \quad (4.2)$$

$$\frac{\partial\psi(0,z)}{\partial x} = \frac{\partial\psi(l,z)}{\partial x} = 0, z \in (0,L) \quad (4.3)$$

Here where $i=\sqrt{-1}$; $a_0>0$ is a given number, a_2 is a complex number and satisfies the following conditions:

$$a_2=\mathrm{Re}a_2+i\mathrm{Im}a_2, \mathrm{Im}a_2>0, \mathrm{Re}a_2<0, \mathrm{Im}a_2\geqslant 2|\mathrm{Re}a_2| \quad (4.4)$$

$a(x), a_1(x,z), v_0(z), v_1(z)$ are real-valued measurable functions and satisfy the following conditions:

$$\mu_0\leqslant a(x)\leqslant \mu_1, \left|\frac{da(x)}{dx}\right|\leqslant \mu_2, \left|\frac{d^2a(x)}{dx^2}\right|\leqslant \mu_3, \forall\forall x\in(0,l), \mu_0,\mu_1,\mu_2,\mu_3= \text{ const } >0 (4.5)$$

$$|a_1(x,z)|\leqslant \mu_4, \left|\frac{\partial a_1(x,z)}{\partial x}\right|\leqslant \mu_5, \left|\frac{\partial^2 a_1(x,z)}{\partial x^2}\right|\leqslant \mu_6, \overset{0}{\forall}(x,z)\in\Omega \quad (4.5)$$

4.6) (4.6)

$$a_1(0,z)=a_1(l,z)=0, z\in(0,L)\mu_3,\mu_4,\mu_5,\mu_6= \text{ const } >0 \quad (4.7)$$

$$4.7)\ |v_s(z)|\leqslant b_s, s=0{,}1, \overset{0}{\forall} z\in(0,L), b_0,b_1= \text{ const } >0$$

$\varphi(x), f(x,z)$ - are complex-valued measurable functions satisfying the following conditions:

$$\varphi\in W_2^2(0,l), \frac{d\varphi(0)}{dx}=\frac{d\varphi(l)}{dx}=0 \quad (4.8)$$

$$f\in W_2^{2,0}(\Omega), \frac{\partial f(0,z)}{\partial x}=\frac{\partial f(l,z)}{\partial x}=0, z\in(0,L) \quad (4.9)$$

Definition 4.1. The solution of the initial boundary value problem (4.1) - (4.3) is defined as the equation (4.1) for almost $(x,z)\in\Omega$, the initial value condition (4.2) for almost $x\in(0,l)$, and the boundary value condition (4.3) for almost $z\in(0,L)$, we will understand the function $\psi=\psi(x,z)$.

In this section we prove the following theorem which shows the existence and uniqueness of the solution of the second kind of initial boundary value problem (4.1) - (4.3).

Theorem 4.2. Suppose that the functions $a(x), a_1(x,t), \varphi(x), f(x,t)$ of the complex number a_2 satisfy the conditions (4.4) - (4.9). Then the initial boundary value problem (4.1) - (4.3) has only one immediate solution in the space $W_2^{2,1}(\Omega)$ and the following estimate holds:

$$\|\psi\|^2_{W_2^{2,1}(\Omega)}\leq c_{29}\left(\|\varphi\|^2_{W_2^2(0,l)}+\|f\|^2_{W_2^{2,0}(\Omega)}+\|\varphi\|^6_{W_2^1(0,l)}+\|f\|^6_{W_2^{1,0}(\Omega)}\right) \quad (4.10)$$

Where $c_{29} > 0$ is a constant.

The proof of this theorem is equivalent to the proof of Theorem 3.1. For the proof of this theorem, the basis functions are the functions from the space of functions $W_2^2(0,l)$ and orthonormal in the space $L_2(0,l)$ and

$$\Lambda X(x) = \lambda X(x), x \in (0,l), \frac{dX(0)}{dx} = \frac{dX(l)}{dx} = 0 \tag{4.11}$$

The system of functions $u_k = u_k = u_k(x), k = 1,2, \ldots$, corresponding to the solutions of the eigenvalue problem $\lambda = \lambda_k, k = 1,2, \ldots$ is used. Here the operator Λ is defined by the formula (3.3).

Acknowledgement

The author thanks the referees and Editor for their valuable comments.

Conflict of interest

The authors declare no conflict of interest.

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