Kadirli Uygulamalı Bilimler Fakültesi Dergisi
Cilt 2, Sayı 1, 1-14, 2022

Journal of Kadirli Faculty of Applied Sciences
Volume 2, Issue 1, 1-14, 2022

Kadirli Uygulamalı Bilimler Fakültesi Dergisi

Journal of Kadirli Faculty of Applied Sciences

Regularity and Green's Relations on the Semigroup of Full Contraction Mappings with A Restricted Range

Muhammad Mansur ZUBAİRU[1*], Zulyadain Dahiru LAWAN[2]

[1]Department of Mathematical Sciences, Bayero University Kano, PM. Box 3011, Kano Nigeria
[2]Department of Science, School of Continuing Education, Bayero University Kano, P.M.B. 3011, Kano Nigeria

[1]https://orcid.org/0000-0001-5099-5956
[2]https://orcid.org/0000-0002-0605-2819
[*]Corresponding author: mmzubairu.mth@buk.edu.ng

Research Article

Article History:
Received: 19.09.2021
Accepted: 14.03.2022
Published online:03.06.2022

Keywords:
Transformation semigroup
Contraction mappings
Ideals
Restricted range
Green's relations

ABSTRACT

In this paper, we determine when two semigroups of full contraction mappings with restricted range are isomorphic. Furthermore, we give necessary and sufficient conditions for an element in the semigroup to be regular and characterize all the Green's equivalences on the semigroup.

Regülerlik ve Kısıtlamalı Görüntüye Sahip Daraltma Fonksiyonlarının Yarı Grupları Üzerinde Green Denklemleri

Araştırma Makalesi

Makale Tarihçesi:
Geliş tarihi: 19.09.2021
Kabul tarihi:14.03.2022
Online Yayınlanma:03.06.2022

Anahtar Kelimeler:
Dönüşüm yarı grupları
Daraltma tasvirleri
İdealler
Kısıtlamalı görüntü
Green bağıntıları

ÖZ

Bu makalede, kısıtlamalı görüntüye sahip tüm daraltma tasvirlerinin iki yarı grubunun ne zaman izomorf olacaklarını bulduk. Ayrıca, bir yarı grup elemanının regüler olması için gerek ve yeter koşulları verdik ve bir yarı gruptaki bütün Green denkliklerini karakterize ettik.

To Cite: Zubairu MM, Lawan ZD., 2022. Regularity and Green's relations on the semigroup of full contraction mappings with a restricted range. Kadirli Uygulamalı Bilimler Fakültesi Dergisi, 2(1): 1-14.

1. Introduction

Denote $[n]$ to be a finite n chain $\{1, 2,..., n\}$. A map say α which has its domain and range both subsets of $[n]$ is said to be a *transformation* of the set $[n]$. A transformation α which has its domain subset of $[n]$ is said to be *partial*. The collection of all partial transformations on $[n]$ is known as the semigroup of partial transformations and is usually denoted by P_n. A partial transformation whose domain is equal to $[n]$ is known as the *full* (or *total*) transformation. The collection of all full transformations on $[n]$ is known as the semigroup of full transformations, which is usually denoted by $T_{n.}$ The algebraic and combinatorial properties of the semigroups P_n and T_n have been extensively studied over the years, see for example (Howie, 1966; Howie et al., 1988; Garba, 1990; Ganyushkin and Mazorchuk, 2009).

A map $\alpha \in T_n$ is said to be a *contraction* if for all $x, y \in [n]$, $|x\alpha - y\alpha| \leq |x - y|$. The collection of all full contraction maps is known as the semigroup of full contraction maps, and is usually denoted by

$$CT_n = \{\alpha \in T_n : \text{for all } x, y \in [n], |x\alpha - y\alpha| \leq |x - y|\}. \qquad (1)$$

In 2013, Umar and Alkharousi (2012) proposed the study of the semigroups of contraction maps on a finite n chain. In this proposal, notations of these semigroups and their various subsemigroups were given. We shall adopt the same notations in this paper. Let Y be a non empty subset of $[n]$. Denote $T([n], Y)$ to be the collection of all $\alpha \in T_n$ such that $[n]\alpha \subseteq Y$. i.e.,

$$T([n], Y) = \{\alpha \in T_n : [n]\alpha \subseteq Y\}.$$

The collection $T([n], Y)$ is known as the semigroup of transformation with restricted range with the usual composition of functions. The algebraic properties as well as the combinatorial properties of the semigroup $T([n], Y)$ have been studied extensively by various scholars, see for example (Nentthein et al., 1975; Sanwong and Sommanee, 2008; Sanwong, 2011; Lei, 2013; Sommanee and Sanwong, 2013). Symons (1975) was the first to introduce and study the semigroup $T([n], Y)$. He described all its automorphisms and determined when the semigroup $T([n], Y_1)$ is isormophic to $T([n], Y_2)$ for $Y_1, Y_2 \subseteq [n]$. In general, the semigroup $T([n], Y)$ is not regular, as such the need to characterize its regular elements. Nenthein et al. (2005) gave a characterization for the regular elements of $T([n], Y)$ and obtained the number of regular elements in $T([n], Y)$. Sanwong and Sommanee (2008) gave a necessary and sufficient conditions for the semigroup $T([n], Y)$ to be regular. In the case that $T([n], Y)$ is not regular, they obtained its largest regular subsemigroup as:

$$F([n], Y) = \{\alpha \in T([n], Y): [n]\alpha = Y\alpha\}.$$

Moreover, they characterized all the Green's equivalences on T ([n], Y) and obtained its maximal inverse subsemigroup. The effect of characterizing the Green's equivalences on a semigroup, is to sort-out the elements of the semigroup. For proper understanding of Green's equivalences, we refer the reader to Howie (1995). Later, Sanwong et al. (2009) described all the maximal and minimal congruences on T ([n], Y). In 2011, Mendes-Goncalves and

Sullivan (2011) obtained all the ideals of T ([n], Y). Sanwong (2011) shows that every regular semigroup S can be embedded in F(S^1, S) (where F((S^1, S) denote the largest regular semigroup in T(S^1, S), for an arbitrary semigroup S). Furthermore, he obtained the characterization of Green's relations and ideals of F([n], Y) when Y is a nonempty finite subset of [n]. The rank of the semigroup T ([n], Y) was computed by Fernandes and Sanwong in 2014. Earlier, Sullivan (2008) took the semigroup which consist of all linear transformations from a vector space V into a fixed subspace W of V and characterized its Green's relations and ideals. Lei (2013) showed that T ([n], Y) is a right abundant semigroup but not left abundant whenever Y is a proper subset of [n].

Let Y be a nonempty subset of $[n]$ and CT_n be as defined in equation (1). Write

$$CT\,([n], Y) = \{\alpha \in CT_n : [n]\alpha \subseteq Y\}. \qquad (2)$$

Notice that for all $\alpha, \beta \in CT([n], Y)$, Im $\alpha\beta \subseteq$ Im $\beta \subseteq Y$ as such $CT([n], Y)$ is a subsemigroup of CT_n. Notice also that if $Y = [n]$, then $CT([n], Y) = CT_n$.

In this paper, we consider the subsemigroup $CT([n], Y)$ and study some of its algebraic properties. In section 1, we give introduction and in section 2, we give the basic definitions needed in subsequent sections, and for proper understanding of the content of the paper. In section 3, we investigate when $CT([n], Y_1)$ is isormophic to $CT\,([n], Y_2)$ for $Y_1, Y_2, \subseteq [n]$ and moreover we show that $CT([n], Y)$ is the union of its left ideals when Y is totally non convex. In section 4, we give complete characterization of regular elements of $CT([n], Y)$. Moreover, we deduce a characterization for regularity for the semigroup $CT([n], Y)$. In section 5, we characterize all the Green's equivalence on $CT([n], Y)$.

2. Definitions and Notations

Let $\alpha \in CT([n], Y)$. Denote the image set of α and |Im α|, respectively by Im α, $h(\alpha)$. For $\alpha, \beta \in CT([n], Y)$, we shall write the composition of α and β as $x(\alpha o \beta) = ((x)\alpha)\beta$ for all $x \in [n]$. A subset Y of $[n]$ is said to be *convex* if $x \leq y$ (for all $x, y \in Y$) and if there exists $z \in [n]$ such that $x < z < y$ implies $z \in Y$, and Y is said to be *non-convex* if it is not a convex subset of $[n]$. A subset B of a set Y is said to be a *sub-convex* subset of Y if B is convex. A subset Y of $[n]$ of order greater than or equal to 2 is said to be *totally non-convex* if Y is non-convex and there is no sub-convex subset of Y say B whose order is greater than or equal to 2.

For example, consider Y_1 = {1, 2, 3} ⊆ {1, 2, 3, 4, 5} = [5], Y_2 = {1, 4, 5} ⊆ [5] and Y_3 = {1, 3, 5} ⊆ (Green, 1951). It is easy to verify that Y_1 is convex and {1, 2}, {2, 3} are both sub-convex subsets of Y_1. However, Y_2 is non-convex and {4, 5} is a sub-convex subset of Y_2.

Moreover, $Y_3$is totally non-convex, since it has no sub-convex subset of order greater than or equal to 2. However, $\{1\}$, $\{3\}$ and $\{5\}$ are sub-convex subsets of Y_3, each of order 1.

Remark 2.0. It is worth noting that, every totally non-convex subset of $[n]$ is non-convex, but the converse is not necessarily true.

Let $\alpha \in CT([n], Y)$, we shall write α in block notation as:

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ x+1 & x+2 & \cdots & x+p \end{pmatrix} \quad (1 \le p \le n), \qquad (3)$$

where Im $\alpha = \{x + 1, x + 2, x + 3, . . . , x + p\} \subseteq Y$ and $(x + i)\alpha^{-1} = A_i$ $(1 \le i \le p)$ are equivalence classes under the relation ker $\alpha = \{(x, y) \in [n] \times [n] : x\alpha = y\alpha\}$. The collection of all the equivalence classes of the relation ker α, is the partition of $[n]$ usually denoted by **Ker** α, i.e., **Ker** $\alpha = \{A_1, A_2, . . . , A_p\}$ and $[n] = A_1 \cup A_2 \cup \cdots \cup A_p$ $(p \le n)$. A subset T_α of $[n]$ is said to be a *transversal* of the partition **Ker** α if $|T_\alpha| = p$ and $|A_i \cap T_\alpha| = 1$, $(1 \le i \le p)$. A transversal T_α is said to be *admissible* if for every x_i , $x_j \in T_\alpha = \{x_i : x_i \in A_i, 1 \le i \le p\}$, $|x_i - x_j| \le |a_i - a_j|$ for all $a_i \in A_i$, $a_j \in A_j$ $(i, j \in \{1, 2, . . . , p\})$ (see (Umar and Zubairu, 2018). A partition **Ker** γ (for $\gamma \in CT([n], Y)$) is said to be a *refinement* of the partition **Ker** α if ker $\gamma \subseteq$ ker α (see Umar and Zubairu, 2018). Thus, if **Ker** $\gamma = \{A_1^{'}, A_2^{'}, . . . , A_p^{'}\}$ and **Ker** $\alpha = \{A_1, A_2, . . . , A_p\}$, then $p \le s$. A map $\alpha \in CT([n], Y)$ is said to be an *isometry* if and only if $|x\alpha - y\alpha| = |x - y|$ for all $x, y \in [n]$. If we consider α as expressed in equation (3), then α is an isometry if and only if $|(x + i) - (x + j)| = |a_i - a_j|$ for all $a_i \in A_i$ and $a_j \in A_j$ $(i, j \in \{1, 2, . . . , p\})$. In other words, α is an isometry if and only if for all $1 \le i \le p$, $a_i \mapsto x_i + e$ for some integer e (called a *translation*) or $a_i \mapsto x_{p-i+1} + e$ for some integer e (called a *reflection*) (Umar and Zubairu, 2018). An element a in a semigroup S is said to be an *idempotent* if and only if $a^2 = a$. A semigroup S is said to be *simple* if S has no ideals other than itself. For basic concept in semigroup theory, we refer the reader to Higgins (1972); Howie (1995); Ganyushkin and Mazorchuk (2009).

3. Isomorphism properties and ideals

In this section, we investigate when two semigroups say $CT([n], Y_1)$ and $CT([n], Y_2)$ are isomorphic for nonempty subsets Y_1 and Y_2 of $[n]$.

Theorem 3.1. Let Y_1, Y_2 be non-empty subsets of $[n]$. Then $CT([n], Y_1)$ and $CT([n], Y_2)$ are isomorphic if and only if there exists an isometry from Y_1 to Y_2.

Proof: Suppose there exists an isomorphism φ: $CT([n], Y_1) \rightarrow CT([n], Y_2)$. Assume $\alpha\varphi = \beta$, for some $\alpha \in CT([n], Y_1)$ and $\beta \in CT(([n], Y_2)$. Since φ is an isomorphism, then $|\text{Im } \alpha| = |\text{Im } \beta|$. Notice that Im α and Im β are convex say Im α = $\{a+1, a+2, ..., a+p\} \subseteq Y_1$ and Im β = $\{b+1, b+2, ..., b+p\} \subseteq Y_2$ for some $a, b \in \mathbb{Z}$. In particular, Im $\alpha = Y_1$ and Im $\beta = Y_2$. Then, the map defined by $a + i \longmapsto b + i$ is a translation and the map defined by $a + i \longmapsto b + (p + i - 1)$ is a reflection, as required.

Conversely, suppose there exists an isometry from Y_1 to Y_2. If the isometry is a translation, $a + i \longmapsto b + i$, then for $\alpha \in CT([n], Y_1)$, let α': $[n] \rightarrow [n]$ be define by

$$x\alpha' = b + i \text{ whenever } x\alpha = a + i.$$

Notice that, for $x, y \in [n]$;

$|x\alpha' - y\alpha'| = |(b + i) - (b + j)| = |i - j| \qquad = |a + i - (a + j)| = |x\alpha - y\alpha| \leq |x - y|$.

Moreover, it is easy to see that Im $\alpha' = \{b + 1, \dots, b + p\} \subseteq Y$. Therefore, $\alpha' \in CT([n], Y_2)$.

Thus, the map φ: $CT([n], Y_1) \rightarrow CT([n], Y_2)$ define by $\alpha\varphi = \alpha'$ is well define and satisfy the property $(\alpha\beta)' = \alpha'\beta'$. To see this, let $\alpha, \beta \in CT([n], Y_1)$ and $x \in [n]$ such that $\alpha = \beta$. Thus $x\alpha = x\beta$ implies $a + i = a + j$. $i = j$ and therefore $b + i = b + j$, which implies $\alpha' = \beta'$, $\alpha\varphi = \beta\varphi$.

To prove the property $(\alpha\beta)' = \alpha'\beta'$, if

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a+1 & a+2 & \cdots & a+p \end{pmatrix}, \ \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_k \\ b+1 & b+2 & \cdots & b+k \end{pmatrix} \in CT([n], Y_1). \text{ Then}$$

$$\alpha\beta = \begin{pmatrix} C_i & C_{i+1} & \cdots & C_{i+r} \\ a+i & a+i+1 & \cdots & a+i+r \end{pmatrix} \in CT([n], Y_1).$$

Now, $\alpha' = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ b+1 & b+2 & \cdots & b+p \end{pmatrix}$ and $\beta' = \begin{pmatrix} B_1 & B_2 & \cdots & B_k \\ b+1 & b+2 & \cdots & b+k \end{pmatrix} \in CT([n], Y_2)$. Thus

$$\alpha'\beta' = \begin{pmatrix} C_i & C_{i+1} & \cdots & C_{i+r} \\ b+i & b+i+1 & \cdots & b+i+r \end{pmatrix} \in CT([n], Y_2).$$

Therefore $(\alpha\beta)\ \varphi = \begin{pmatrix} C_i & C_{i+1} & \cdots & C_{i+r} \\ b+i & b+i+1 & \cdots & b+i+r \end{pmatrix} = \alpha'\beta'$. It now follows that for $\alpha, \beta \in CT([n], Y_1)$ we have $(\alpha\beta)\varphi = \varphi\alpha\varphi\beta$. φ is a homomorphism. One can easily show that φ is a bijection and hence it is an isomorphism.

Now, if the isometry is a reflection, i.e., $a+i \longmapsto (b+p+i-1)$. Then for $\alpha \in CT$ $([n], Y_1)$, let $\alpha': [n] \to [n]$ be define by $x\alpha' = b + (p+i-1)$ whenever $x\alpha = a+i$. Then, the map φ: $CT([n], Y_1) \to CT([n], Y_2)$ defined by $\alpha\varphi = \alpha'$ is well-defined and satisfy the property $(\alpha\beta)' = \alpha'\beta'$. It is easy to see (as in the previous paragraph) that φ is an isomorphism. Hence, the proof.

The next results show that the semigroup $CT([n], Y)$ can be express as a disjoint union of left ideals. We begin our investigation with the following lemma.

Lemma 3.2. Let $Y \subseteq [n]$ be a disjoint union of nonempty convex subsets $B_i (i \in \{1,2,3,\cdots,p \leq n\})$ satisfying the following conditions:

1. $B_i \cap B_j = \emptyset$ if and only if $i \neq j$;
2. $B_1 < B_2 < B_3 < \cdots < B_p$;
3. $\max(B_i) + 2 \leq \min(B_{i+1})$.

Then, each $CT([n], B_i)$ $(i \in \{1,2,3,\cdots,p \leq n\})$ is a left ideal of $CT([n], Y)$.

Proof. Notice that each B_i $(i \in \{1,2,3,\cdots,p \leq n\})$ is nonempty. Thus for $x \in B_i$, the map

$$\alpha_x = \begin{pmatrix} [n] \\ x \end{pmatrix} \in CT([n], B_i)$$

and so each $CT([n], B_i) \neq \emptyset$. Now, let $\alpha \in CT([n], B_i)$ for $1 \leq i \leq p$ and $\beta \in CT$([n], Y), then

$$[n]\beta\alpha = ([n]\beta)\alpha \subseteq Y\alpha \subseteq B_i.$$

Thus, $\beta\alpha \in CT([n], B_i)$, as required.

As a consequences, we have the following theorem.

Theorem 3.3. Let $Y \subseteq [n]$ be a disjoint union of nonempty convex subsets $B_i (i \in \{1,2,3,\ldots,p\},\ p \leq n)$ satisfying the following conditions:

1. $B_i \cap B_j = \emptyset$ if and only if $i \neq j$;
2. $B_1 < B_2 < B_3 < \cdots < B_p$;
3. $\max(B_i) + 2 \leq \min(B_{i+1})$.

Then, $1 \leq \min(B_1)$, $\max(B_p) \leq n$ and $CT([n], Y) = \bigcup_{i=1}^{p} CT([n], B_i)$.

Proof. Notice that by Lemma 3.2, $CT([n], B_i)$ is a left ideal of $CT([n], Y)$ for all $1 \leq i \leq p$. Thus it is now the case of showing two sets are equal.

Now let $\alpha \in CT([n], Y)$. Suppose, Im $\alpha \subseteq Y$, then there exists $1 \leq i \leq p$ such that $B_i =$ Im α. Thus, Im $\alpha \subseteq B_i$, $\alpha \in CT([n], B_i) \subseteq \bigcup_{i=1}^{p} CT([n], B_i)$, $\alpha \in \bigcup_{i=1}^{p} CT([n], B_i)$. Thus

$$CT([n], Y) \subseteq \bigcup_{i=1}^{p} CT([n], B_i). \qquad \text{(i)}$$

Now, let $\alpha \in \bigcup_{i=1}^{p} CT([n], B_i)$. Notice that $B_i \cap B_j = \emptyset$ for all $i \neq j$. Thus, there exists $i \in \{1, \cdots, p\}$ such that $\alpha \in CT([n], B_i)$, $\text{Im } \alpha \subseteq B_i \subseteq Y$, $\text{Im } \alpha \subseteq Y$. Therefore, $\alpha \in CT([n], Y)$. Thus

$$\bigcup_{i=1}^{p} CT([n], B_i) \subseteq CT([n], Y). \qquad \text{(ii)}$$

Hence by equation (i) and (ii) we have $\bigcup_{i=1}^{p} CT([n], B_i) \subseteq CT([n], Y)$, as required.

If Y is convex then $CT([n], Y)$ can not be expressed as a union of its left ideal as in the remark below.

Remark 3.4. If Y is convex, then Y does not satisfy the condition (3) of Lemma 3.2. Thus, $CT([n],Y)$ can not be expressed as a union of $CT([n], B_i)(1 \leq i \leq \text{p})$.

4. Regularity in the semigroup $CT([n], Y)$

An element a in a semigroup S is regular if there exists $b \in S$ such that $a = aba$. A semigroup S is said to be regular if every element of S is regular. For an arbitrary semigroup S, we shall denote the set of regular elements of S by $Reg\,(S)$.

In Nentthein et al. (2005), Nenthein et al., characterized the regular elements of the semigroup $T([n], Y)$ as in the following lemma:

Lemma 4.1. (Nentthein et al. (2005), Theorem 2.1). For $\alpha \in T([n], Y)$, the following statements are equivalent:

(i) $\alpha \in RegT([n], Y)$;

(ii) $ran\ \alpha = Y\alpha$;

(iii) $x \ker \alpha \cap Y \neq \emptyset$ for every $x \in [n]$;

(iv) $x\alpha^{-1} \cap Y \neq \emptyset$ for every $x \in ran\ \alpha$.

It is worth noting that the characterization given above by Nenthein *et al.*, does not hold for the semigroup $CT([n], Y)$. To see this, consider $[n] = \{1, 2, \ldots, 6\}$, $Y = \{1, 2, 3, 4, 5\}$ and choose α *as:* $\alpha = \begin{pmatrix} 1 & \{2,3\} & \{4,6\} & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$. Now as in the above lemma, α satisfy condition (ii), thus α is regular in $T([n], Y)$. However, one can easily verify that no β in CT (Hall, 1982, Y) that satisfy $\alpha = \alpha\, \beta\, \alpha$. Hence, α is not regular in $CT([n], Y)$. Thus, there is need to come up with a characterization for the regular elements in $CT([n], Y)$.

It is well known that CT_n is regular for $1 \leq n \leq 3$ but not regular for $n \geq 4$ (see Umar and Zubairu, 2018). Moreover, it is easy to verify that $CT([n], Y) = CT_n$ for $1 \leq n \leq 2$. However, the semigroup $CT([n], Y)$ is not regular for $n \geq 3$ as we shall see in this section.

The following lemmas and remark are found useful in our subsequent discussion.

Lemma 4.2. (Fernandes and Sanwong, 2014, Lemma 1.2)**.** *Let* $\alpha \in CT_n$ *and let* $|\text{Im } \alpha| = p$. *Then* Im α *is convex.*

Lemma 4.3. (Umar and Zubairu, 2018, Corollary 1.13)**.** *Let* $\alpha \in CT_n$. *Then* α *is regular if and only if* **Ker** α *has a convex transversal.*

Remark 4.4. (Umar and Zubairu, 2018, Remark 1.14)**.** *A transversal* T_α *of* **Ker** α *(*$\alpha \in CT_n$*) is admissible if and only if* T_α *is convex.*

Next, we now characterize the regular elements of $CT([n], Y)$ in the theorem below.

Theorem 4.5. *Let* $CT([n], Y)$ be as defined in equation (1) and let $\alpha \in CT([n], Y)$ *be as expressed in equation* (3). *Then* α *is regular if and only if* **Ker** α *has a convex transversal* $T_\alpha \subseteq Y$.

Proof. Let $\alpha \in CT([n], Y)$ be regular. Notice that $\alpha \in CT_n$. Thus by Lemma 4.3, α has a convex transversal say $T_\alpha = \{t + 1, t + 2, \ldots, t + p\}$ for $t \in \mathbb{Z}$. Notice that and T_α is convex, thus by Remark 4.4, T_α is admissible in CT_n. But for T_α to be admissible in $CT([n], Y)$, T_α must be a
subset of Y.

Conversely, suppose **Ker** α has a convex transversal $T_\alpha \subseteq Y$ say $T_\alpha = \{t+1, t+2, \ldots, t+p\}$ and since $\alpha \in CT([n], Y)$ we may (without loss of generality) write α as:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ s+1 & s+2 & \cdots & s+p \end{pmatrix},$$

where $\{s + 1, s + 2, \ldots, s + p\ p\ 1, s + p\} \subseteq Y$. Now define β as:

$$\beta = \begin{pmatrix} \{1,2,\ldots,s+1\} & s+2 & \cdots s+p-1 & \{s+p,\ldots,n\} \\ t+1 & t+2 & \cdots t+p-1 & t+p \end{pmatrix}.$$

It is clear that β is a contraction and since Im $\beta = T_\alpha \subseteq Y$, then $\beta \in CT([n], Y)$. It follows easily that $\alpha\beta\alpha = \alpha$.

We now prove the following result, which give a necessary and sufficient condition for the semigroup $CT([n], Y)$ to be regular.

Theorem 4.6. *The semigroup* $CT([n], Y)$ *is regular if and only if* Y *is totally non-convex subset of* $[n]$.

Proof. Let $CT([n], Y)$ be a regular semigroup (i.e., **Ker** α has a convex transversal say $T_\alpha \subseteq Y$ for all $\alpha \in CT([n], Y)$). Suppose by way of contradiction that Y is either convex or non-convex.

Case 1. Suppose Y is convex. Let $Y = \{a + 1, \ldots, a + r\} \subseteq [n]$ (for some $r \geq 2$). Choose

$$\alpha = \begin{pmatrix} \{1,\ldots,a+1,\ldots a+r\} & \{a+r+1,\ldots,n\} \\ a+1 & a+2 \end{pmatrix}.$$

Then, clearly $\alpha \in CT([n], Y)$ is of rank 2 and also $T_\alpha = \{a+r, a+r+1\}$ is a convex transversal of **Ker** α. Notice that T_α is not a subset of Y. Therefore by Theorem 4.5, α is not regular, a contradiction.

Case 2. Suppose Y is non-convex. This implies there exists a sub-convex subset of Y of order greater than or equal to 2, thus, the results follows from Case 1.

Conversely, if Y is totally non-convex subset of $[n]$. Then $CT([n], Y) = \left\{ \begin{pmatrix} [n] \\ a \end{pmatrix} : a \in Y \right\}$.

Notice that each element in $CT([n], Y)$ is an idempotent and as such regular, as required.

We now have the following corollary.

Corollary 4.7. *If* $1 < |Y| < n$ *and Y has of sub-convex subset of order greater than 1. Then the semigroup* $CT([n], Y)$ *is not regular.*

Proof. Suppose by way of contradiction that $CT([n], Y)$ is regular. Let $x, y \in Y$ be such that $x \neq y$. Let $\alpha \in CT([n], Y)$ be define as $\begin{pmatrix} A_1 & A_2 \\ x & y \end{pmatrix}$ and choose $c \in [n] \setminus Y$ such that $c = \max(x\alpha^{-1})$.

Thus by Theorem 4.5, α must have a convex transversal say T_α subset of Y, but clearly, $c \in T_\alpha$ which contradicts the fact that $T_\alpha \subseteq Y$. The results follow.

As a consequence we readily have the following result.

Corollary 4.8. *The semigroup* $CT([n], Y)$ *is not regular for all* $n \geq 3$.

Proof. Let

$$\alpha = \begin{pmatrix} \{1,2\} & 3 & \cdots & n \\ 1 & 2 & \cdots & 2 \end{pmatrix} \in CT([n], \{1,2\}).$$

Notice that $T_\alpha = \{2,3,4,\ldots,n\} \nsubseteq Y$. Therefore, by Theorem 4.5, α is not regular, as required.

Corollary 4.9. *If Y is totally non-convex subset of* $[n]$. *Then each* $\alpha \in CT([n],Y)$ *is an idempotent of rank 1.*

Proof. Notice that each element in $CT([n], Y)$ is a constant map of height 1 and as such is an idempotent of rank 1.

Product of idempotents is not necessary an idempotent as demonstrated in the example below.

Example 4.10. *In the semigroup* $CT([n], Y)$, *the product of idempotents is not necessary an idempotent. To see this, let* (Howie, 1966) $= \{1,\ldots,9\}$, $Y = \{1, 2, 3, 7, 8, 9\}$ *and choose* $\alpha = \begin{pmatrix} 1 & 2 & \{3,\ldots,9\} \\ 1 & 2 & 3 \end{pmatrix}$ *and* $\beta = \begin{pmatrix} \{1,5,9\} & \{2,4,6,8\} & \{3,7\} \\ 9 & 8 & 7 \end{pmatrix}$ *elements of CT* (Howie, 1966,

Y). *Then clearly* α *and* β *are idempotents in CT* (Howie, 1966, Y). *The products* $\alpha\beta = \begin{pmatrix} 1 & 2 & \{3,\ldots,9\} \\ 9 & 8 & 7 \end{pmatrix}$ *is not idempotent.*

Proposition 4.11. *If Y is totally non-convex subset of* $[n]$. *Then the semigroup* $CT([n], Y)$ *is simple.*

Proof. The result follows since each element in $CT([n], Y)$ is of rank 1.

Theorem 4.12. *Suppose Y is totally non-convex subset of* $[n]$. *If* $|Y| = r$, *then* $|CT([n], Y)| = r$.

Proof. Since Y has no sub-convex subset of order greater than or equal to 2, then $CT([n], Y)$ contains element of rank 1 and obviously there are r of them.

Remark 4.13. It is worth noting from the proceeding results that, the semigroup $CT([n], Y)$ is regular if Y is totally non-convex subset of $[n]$, otherwise $CT([n], Y)$ is not regular.

5. Green's relations on the semigroup $CT([n], Y)$

Let S be a semigroup without identity element and S^1 be a monoid. The five equivalence relations on S known as Green's relations were first introduced by J. A. Green's in 1995. The primary aim of defining these relations is to study the structure of a semigroup S. These relations are defined as follows. For $a, b \in S$, $a \mathcal{L} b$ if and only if $S^1a = S^1b$ (i.e., a and b generates the same principal left ideal, here a and b are said to be $\mathcal{L}$ related); $a \mathcal{R} b$ if and only if $aS^1 = bS^1$ (i.e., a and b generates the same principal right ideal, here a and b are said to be $\mathcal{R}$ related); $a \mathcal{J} b$ if and if $S^1aS^1 = S^1 bS^1$ (a and b generate the same principal two sided ideal, in this case, a and b are said to be $\mathcal{J}$ related).The relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ while the relation $\mathcal{D}$ is a join of the relations $\mathcal{L}$ and $\mathcal{R}$ i.e., $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. These relations are all equivalences on S. For more details on Green's relations we refer the reader to Green (1951); Higgins (1992); Howie (1995); Ganyushkin and Mazorchuk (2009). The Green's relations for the semigroup CT_n and some of its subsemigroups have been investigated in Umar and Zubairu (2018). Here, we also deduce the characterizations for the Green's relations on the semigroup $CT([n], Y)$. Throughout this section, we will consider $1 < |Y| < n$.

Now denote

$$\alpha = \begin{pmatrix} A_1 & A_2 \cdots & A_p \\ x_1 & x_2 \cdots & x_p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 \cdots & B_p \\ y_1 & y_2 \cdots & y_p \end{pmatrix} \qquad (1 \le p \le n). \qquad (5)$$

Before we begin our investigation, we first note the following results from Umar and Zubairu (2018) which are found to be useful in what to follows.

Theorem 5.1. (Umar and Zubairu, 2021), Corollary 5.3)**.** *Let* $\alpha, \beta \in CT_n$ *be as expressed in equation* (5). *Then*

(i) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* **Ker** α *and* **Ker** β *have convex refinement partitions,* **Ker** γ_1 *and* **Ker** γ_2 *(for some* γ_1 *and* γ_2 *in* CT_n*), respectively, such that there exists either a translation* $\tau_i \mapsto \sigma_i$ *satisfying* $\tau_i\alpha = \sigma_i\beta$ *or a reflection* $\tau_i \mapsto \sigma_{s-i+1}$ *satisfying* $\tau_i\alpha = \sigma_{s-i+1}\beta$ *for all* $i = 1, \ldots, s$ $(s \geq p)$*, where* $T\gamma_1 = \{\tau_1, \ldots, \tau_s\}$ *and* $T\gamma_2 = \{\sigma_1, \ldots, \sigma_s\}$*, are the convex transversals of* **Ker** γ_1 *and* **Ker** γ_2*, respectively;*

(ii) $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $\ker \alpha = \ker \beta$*;*

(iii) $(\alpha, \beta) \in \mathcal{H}$ *if and only if* $\ker \alpha = \ker \beta$ *and* **Ker** α *and* **Ker** β *have convex refinement partitions,* **Ker** γ_1 *and* **Ker** γ_2 *(for some* γ_1 *and* γ_2 *in* CT_n*), respectively, such that there exists either a translation* $\tau_i \mapsto \sigma_i$ *satisfying* $\tau_i\alpha = \sigma_i\beta$ *or a reflection* $\tau_i \mapsto \sigma_{s-i+1}$ *satisfying* $\tau_i\alpha = \sigma_{s-i+1}\beta$ *for all* $i = 1, \ldots, s$ $(s \geq p)$*, where* $T\gamma_1 = \{\tau_1, \ldots, \tau_s\}$ *and* $T\gamma_2 = \{\sigma_1, \ldots, \sigma_s\}$*, are the convex transversals of* **Ker** γ_1 *and* **Ker** γ_2*, respectively;*

(iv) $(\alpha, \beta) \in \mathcal{D}$ *if and only if there exist isometries* $\vartheta 1$ *and* $\vartheta 2$ *from* **Ker** γ_1 *to* **Ker** γ_2 *and from* Im α *to* Im β*, respectively.*

We now characterize the Green's relations on the semigroup $CT([n], Y)$.

Theorem 5.2. *Let* $\alpha, \beta \in CT([n], Y)$*. Then,* $\alpha \mathcal{L} \beta$ *if and only if there exist refinements* **Ker** γ_1, **Ker** γ_2 *(for some* $\gamma_1, \gamma_2 \in CT([n], Y)$*) of* **Ker** α *and* **Ker** β *respectively, such that* **Ker** γ_1, **Ker** γ_2 *have admissible transversals* $T\gamma_1 = \{\tau_1, \tau_2, \ldots, \tau_s\}$, $T\gamma_2 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ *both subset of* Y *or* $T\gamma_1 = T\gamma_2 = [n]$ *with the property that there exists either a translation* $\tau_i \mapsto \sigma_i$ *satisfying* $\tau_i\alpha = \sigma_i\beta$ *or a reflection* $\tau_i \mapsto \sigma_{s-i+1}$ *satisfying* $\tau_i\alpha = \sigma_{s-i+1}\beta$ *for all* $i = 1, \ldots, s$ $(s \geq p)$.

Proof. Let $\alpha, \beta \in CT([n], Y)$ be such that $\alpha \mathcal{L} \beta$. Then there exist $\gamma_1, \gamma_2 \in CT([n], Y)^1$ such that $\alpha = \gamma_1\beta$ and $\beta = \gamma_2\alpha$.

Notice that, Im α, Im $\beta \subseteq Y$, and also $\alpha, \beta \in CT_n$. Thus, by Theorem 5.1, **Ker** α and **Ker** β have refinements partitions say **Ker** γ_1 and **Ker** γ_2, respectively, (for some $\gamma_1, \gamma_2 \in CT_n$) with admissible transversals say $T\gamma_1 = \{\tau_1, \tau_2, \ldots, \tau_s\}$, $T\gamma_2 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ such that there exists either a translation $\tau_i \mapsto \sigma_i$ satisfying $\tau_i\alpha = \sigma_i\beta$ or a reflection $\tau_i \mapsto \sigma_{s-i+1}$ satisfying $\tau_i\alpha = \sigma_{s-i+1}\beta$ for all $i \in \{1, 2, \ldots, s\}$ $(s \geq p)$. Notice that, $T\gamma_1$ and $T\gamma_2$ are admissible, the maps

$$\delta_1 = \begin{pmatrix} A'_1 & A'_2 \cdots & A'_s \\ \tau_1 & \tau_2 \cdots & \tau_s \end{pmatrix} \text{ and } \delta_2 = \begin{pmatrix} B'_1 & B'_2 \cdots & B'_s \\ \sigma_1 & \sigma_2 \cdots & \sigma_s \end{pmatrix} \text{ are in } CT_n .$$

However, for δ_1, δ_2 to be in $CT([n], Y)$, Im δ_1 and Im δ_2 must be subsets of Y (i.e., $T\delta_1 = \{\tau_1, \tau_2, \ldots, \tau_s\} \subseteq Y$ and $T\delta_2 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\} \subseteq Y$) or Im γ_1 = Im $\gamma_2 = [n]$.

Conversely, Suppose there exist refinements **Ker** γ_1 and **Ker** γ_2 (for some $\gamma_1, \gamma_2 \in CT([n], Y)$) of **Ker** α and **Ker** β respectively, such that **Ker** γ_1 and **Ker** γ_2 have admissible transversals $T\gamma_1 = \{\tau_1, \tau_2, \ldots, \tau_s\}$ and $T\gamma_2 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$, respectively, both subset of Y or $T\gamma_1 = T\gamma_2 = [n]$ with the property that there exists either a translation $\tau_i \mapsto \sigma_i$ satisfying $\tau_i\alpha = \sigma_i\beta$ or a reflection $\tau_i \mapsto \sigma_{s-i+1}$ satisfying $\tau_i\alpha = \sigma_{s-i+1}\beta$ for all $i \in \{1, 2, \ldots, s\}$ $(s \geq p)$.

If $T\gamma_1 = T\gamma_2 = [n]$. Then define $\gamma_1 = \gamma_2 = id_{[n]}$. Thus, γ_1, γ_2 are in $CT([n], Y)$ and $\alpha = id_{[n]}\beta = \beta$. Hence, $\alpha \mathcal{L} \beta$.

Now if there is a translation translation $\tau_i \mapsto \sigma_i$ satisfying $\tau_i\alpha = \sigma_i\beta$ (i = 1,. . . ,s). Then define

$\gamma_1 = \begin{pmatrix} A'_1 & A'_2 \cdots & A'_s \\ \sigma_1 & \sigma_2 \cdots & \sigma_s \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} B'_1 & B'_2 \cdots & B'_s \\ \tau_1 & \tau_2 \cdots & \tau_s \end{pmatrix}$. Then it is easy to see that γ_1 and γ_2 are in $CT([n], Y)$.

If there is a reflection $\tau_i \mapsto \sigma_{s-i+1}$ satisfying $\tau_i\alpha = \sigma_{s-i+1}\beta$ for all $i \in \{1, 2, \ldots, s\}$ $(s \geq p)$.

Then define $\gamma_1 = \begin{pmatrix} A'_1 & A'_2 \cdots & A'_s \\ \sigma_s & \sigma_{s-1} \cdots & \sigma_1 \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} B'_1 & B'_2 \cdots & B'_s \\ \tau_s & \tau_{s-1} \cdots & \tau_1 \end{pmatrix}$. Then one can easily show that γ_1 and γ_2 are contractions in $CT([n], Y)$. Hence, $\alpha \mathcal{L} \beta$.

Theorem 5.3. *Let $\alpha, \beta \in CT([n], Y)$. Then $\alpha \mathcal{R} \beta$ if and only if* $\ker \alpha = \ker \beta$.

Proof. Let $\alpha, \beta \in CT([n], Y)$ and suppose $\alpha \mathcal{R} \beta$. This implies that there exist $\gamma_1, \gamma_2 \in CT([n], Y)^1$ such that $\alpha = \beta\gamma_1$ and $\beta = \alpha\gamma_2$. Suppose $(x, y) \in \ker \alpha$. Then $x\beta = x(\alpha\gamma_1) = (x\alpha)\gamma_1 = (y\alpha)\gamma_1 = y(\alpha\gamma_1) = y\beta$. This implies that $\ker \alpha \subseteq \ker \beta$. Similarly, $\ker \beta \subseteq \ker \alpha$. Thus, $\ker \alpha = \ker \beta$ follows easily.

Conversely, suppose $\ker \alpha = \ker \beta$. We may write α and β as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_s \\ a & a+1 & \cdots & a+s-1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_s \\ b & b+1 & \cdots & b+s-1 \end{pmatrix}.$$

Now define $\gamma_1 = \begin{pmatrix} \{1,2,\ldots,b\} & b+1 & \cdots b+s-2 & \{b+s-1,\ldots,n\} \\ a & a+1 & \cdots a+s-2 & a+s-1 \end{pmatrix}$ and

$\gamma_2 = \begin{pmatrix} \{1,2,\ldots,a\} & a+1 & \cdots a+s-2 & \{a+s-1,\ldots,n\} \\ b & b+1 & \cdots b+s-2 & b+s-1 \end{pmatrix}$. Notice that, $\operatorname{Im} \gamma_1 = \operatorname{Im} \alpha$, $\operatorname{Im} \gamma_2 = \operatorname{Im} \beta$ and since $\operatorname{Im} \alpha, \operatorname{Im} \beta \subseteq Y$, we conclude that $\operatorname{Im} \gamma_1, \operatorname{Im} \gamma_2 \subseteq Y$. Therefore, it easily follows that $\gamma_1, \gamma_2 \in CT([n], Y)$. Thus, $\alpha \mathcal{R} \beta$.

Theorem 5.4. *Let $\alpha, \beta \in CT([n], Y)$. Then, $\alpha \mathcal{D} \beta$ if and only if there exist isometries υ_1 and υ_2 from* **Ker** γ_1 *to* **Ker** γ_2 *and from* $\operatorname{Im} \alpha$ *to* $\operatorname{Im} \beta$*, respectively.*

Proof. The results follows easily from Theorem 5.2 and Theorem 5.3.

6. Conclusions

In this paper, we give a necessary and sufficient conditions for two semigroups of full contraction mappings with a restricted range to be isomorphic. Also, we have shown that whenever Y is a union of nonempty convex subsets $B_i (i \in \{1,2,3,\dots,p \leq n\})$ satisfying certain conditions, the semigroup $CT([n], Y)$ can be written as the union of left ideals of $CT([n], Y)$. Further, we characterized the regular elements for the semigroup $CT([n], Y)$, and also investigate the conditions that make the semigroup $CT([n], Y)$ regular. Moreover, we characterized all its Green's equivalences.

Acknowledgement:

We wish to thank Prof. Abdullahi Umar and the anonymous referees for valuable comments and suggestions which help to improve the quality of this paper.

Statement of Conflict of Interest

Authors have declared no conflict of interest.

Author's Contributions

The contribution of the authors is equal.

References

Ali B, Umar A, Zubairu MM., 2018. Regularity and Greens relations for the semigroups of partial and full contractions of a finite chain. https://www.researchgate.net/publication/323598062

Fernandes VH, Sanwong J., 2014. On the rank of semigroups of transformations on a fi set with restricted range. Algebra Colloq, 21: 497-510.

Garba GU., 1990. Idempotents in partial transformation semigroups. Proc. Roy. Soc. Edinb. 116(A): 359-366.

Ganyushkin O, Mazorchuk V., 2009. Classical finite transformation semigroups. Springer-VerlagLondon Limited, 2009. doi:10.1007/978-1-84800-281-4.

Green JA., 1951. On the structure of semigroups. Ann. of Math. 54(2): 163-172.

Hall TE., 1982. Some properties of local subsemigroups inherited by larger subsemigroups. Semigroup Forum, 1982: 35-49.

Higgins PM., 1992. Techniques of semigroup theory. Oxford University Press, 1992.

Howie JM, Robertson EF, Schein BM., 1988. A combinatorial property of finite full transformation semigroups. Proc. Roy. Soc. Edinb. 109A(1988): 319-328.

Howie JM., 1966. The subsemigroup generated by the idempotents of a full transformation semigroup. J. Lond. Math. Soc. 41(1966): 707-716.

Howie JM., 1995. Fundamentals of semigroup theory. Oxford University Press, 1995.

Lei S., 2013. A Note on abundance of certain semigroups of transformations with restricted range. Semigroup Forum, 87(2013): 681-684.

Mendes-Goncalves S, Sullivan RP., 2011. The ideal structure of semigroups of transformations with restricted range. Bull. Aust. Math. Soc. 83(2011): 289-300.

Nentthein S, Youngkhong P, Kemprasit Y., 2005. Regular elements of some transformation semigroups. Pure Math. Appl. 16(3): 307-314.

Sanwong J, Sommanee W., 2008. Regularity and Green's relations on the semigroup of transformations with restricted range. Int. J. Math. Sci. 794013(2008). doi:10.1115/2008/794013.

Sanwong J., 2011. The regular part of a semigroup of transformations with restricted range. Semigroup Forum 83: 134-146.

Sanwong J, Singha B, Sullivan RP., 2009. Maximal and minimal congruence on some semigroups. Acta Math. Sin. 25(3): 455-466.

Sommanee W, Sanwong J., 2013. Rank and idempotent rank of finite full transformation semigroups with restricted range. Semigroup Forum, 87: 230-242.

Sullivan RP., 2008. Semigroups of linear transformations with restricted range. Bull. Aust. Math. Soc. 77(2008): 441-452.

Symons JSV., 1975. Some results concerning a transformation semigroup. J. Aust. Math. Math. Soc. Ser. A, 19(4): 413–425.

Umar A, Al-Kharousi F., 2012. Studies in semigroup of contraction mappings of a finite chain. The Research Council of Oman Research grant proposal No. ORG/CBS/12/007, 6 March, 2012.

Umar A, Zubairu MM., 2018. On certain semigroups of full contractions of a finite chain. arXiv:1804.10057v1[math.GR], 2018.

Umar A, Zubairu MM., 2021. On certain semigroups of contractions mappings of a finite chain. Algebra and Discrete Math. 32(2): 299-320.